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Subadditivity of chance measure

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Abstract

Chance theory is a mathematical methodology for dealing with indeterminacy phenomena involving uncertainty and randomness. In this paper, some properties of chance space are investigated. Based on this, the subadditivity theorem, null-additivity theorem, and asymptotic theorem of chance measure are proved.

Keywords: Uncertainty theory; Chance theory; Chance measure; Subadditivity

Introduction

Uncertainty theory founded by Liu [1] in 2007 is a branch of axiomatic mathematics based on normality, duality, subadditivity, and product axioms. After that, many researchers widely studied the uncertainty theory and made significative progress. Liu [1] presented the concept of uncertain variable and uncertainty distribution. Then, a sufficient and necessary condition of uncertainty distribution was proved by Peng and Iwamura [2] in 2010. In addition, a measure inversion theorem was proposed by Liu [3] from which the uncertain measures of some events can be calculated via the uncertainty distribution. After proposing the concept of independence [4], Liu [3] presented the operational law of uncertain variables. In order to sort uncertain variables, Liu [3] proposed the concept of expected value of uncertain variable. A useful formula was presented by Liu and Ha [5] to calculate the expected values of monotone functions of uncertain variables. Based on the expected value, Liu [1] presented the concepts of variance, moments, and distance of uncertain variables. In order to characterize the uncertainty of uncertain variables, Liu [4] proposed the concept of entropy in 2009. Dai and Chen [6] verified the positive linearity of entropy and presented some formulas for calculating the entropy of monotone function of uncertain variables. Chen and Dai [7] discussed the maximum entropy principle for selecting the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints. In order to make an extension of entropy, Chen et al. [8] proposed a concept of cross-entropy for comparing an uncertainty distribution against a reference uncertainty distribution. Liu [9] introduced a paradox of stochastic finance theory based on uncertainty theory and uncertain differential equation. In addition, an uncertain integral was proposed by Chen and Ralescu [10] presented with respect to the general Liu process.

In 2013, Liu [11] proposed chance theory by giving the concepts of uncertain random variable and chance measure in order to describe the situation that uncertainty and randomness appear in a system. Some related concepts of uncertain random variables such as chance distribution, expected value, and variance were also presented by Liu [11].

As an important contribution to chance theory, Liu [12] presented an operational law of uncertain random variables. After that, uncertain random variables were discussed widely. Yao and Gao [13] provided a law of large numbers for uncertain random variables. Gao and Yao [14] gave some concepts and theorems of uncertain random process. In addition, Yao and Gao [13] proposed an uncertain random process as a generalization of both stochastic process and uncertain process. As applications of chance theory, Liu [12] proposed uncertain random programming. Uncertain random risk analysis was presented by Liu and Ralescu [15]. Besides, chance theory was applied into many fields, and many achievements were obtained, such as uncertain random reliability analysis [16], uncertain random logic [17], uncertain random graph [18], and uncertain random network [18].

In this paper, some properties of chance space are investigated. Based on this, the subadditivity theorem, null-additivity theorem, and asymptotic theorem of chance measure are proposed.

Preliminary

As a branch of axiomatic mathematics, uncertainty theory aims to deal with human uncertainty. In this section, we will provide a brief introduction to uncertain variables and uncertain random variables, which will be used throughout this paper.

Uncertain variables

Definition 1. (Liu [1]) Let Γ be a non-empty set and $\mathcal{L}$ be a σ-algebra on Γ. Each element in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to $[0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.* In 2009, Liu [4] defined product uncertain measure via the fourth axiom of uncertainty theory.

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Then, the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable is a real-valued function on an uncertainty space, which is defined as follows.

Definition 2. (Liu [1]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\xi^{-1}(B) = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

In order to describe uncertain variables, a concept of uncertainty distribution was introduced by Liu [1].

Definition 3. (Liu [1]) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

Definition 4. (Liu [4]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Theorem 1. *(Liu [1]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

To represent the average value of an uncertain variable in the sense of uncertain measure, the expected value is defined as follows.

Definition 5. (Liu [1]) Let ξ be an uncertain variable. Then, the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Definition 6. (Liu [1]) Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x. \quad (1)$$

For calculating the expected value by inverse uncertainty distribution, Liu and Ha [5] proved the following theorem.

Theorem 2. *(Liu and Ha [5]) Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{2}$$

Uncertain random variables

In 2013, Liu [11] first proposed chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, operational law, expected value, and so on. The chance space is referred to the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, in which $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Definition 7. (Liu [11]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then, the chance measure of Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\}\mathrm{d}r.$$

Notation: For a real number r, the set $\Theta_r = \{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r\}$ is a subset of Ω but not necessarily an event in $\mathcal{A}$. In this case, $\Pr\{\Theta_r\}$ is assigned by

$$\Pr\{\Theta_r\} = \begin{cases} \inf\limits_{A \in \mathcal{A}, A \supset \Theta_r} \Pr\{A\}, & \text{if} \inf\limits_{A \in \mathcal{A}, A \supset \Theta_r} \Pr\{A\} < 0.5 \\ \sup\limits_{A \in \mathcal{A}, A \subset \Theta_r} \Pr\{A\}, & \text{if} \sup\limits_{A \in \mathcal{A}, A \subset \Theta_r} \Pr\{A\} > 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{3}$$

Liu [11] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

(a) $\mathrm{Ch}\{\Gamma \times \Omega\} = 1, \mathrm{Ch}\{\emptyset\} = 0$;
(b) $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$ for any event Θ;
(c) $\mathrm{Ch}\{\Theta_1\} \leq \mathrm{Ch}\{\Theta_2\}$ for any event $\Theta_1 \subset \Theta_2$.

First, we give an equivalent definition of $\Pr\{\cdot\}$ in (3).

Lemma 1. *Let $(\Gamma, \mathcal{A}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Denote that $\Theta_B = \{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \in B\}$ for any Borel set B. Then, we have*

$$\Pr\{\Theta_B\} = \inf_{A \in \mathcal{A}, A \supset \Theta_B} \Pr\{A\} \wedge \left(\sup_{A \in \mathcal{A}, A \subset \Theta_B} \Pr\{A\} \vee 0.5 \right) \tag{4}$$

$$\Pr\{\Theta_B\} = \sup_{A \in \mathcal{A}, A \subset \Theta_B} \Pr\{A\} \vee \left(\inf_{A \in \mathcal{A}, A \supset \Theta_B} \Pr\{A\} \wedge 0.5 \right) \tag{5}$$

Proof. The argument breaks down into three cases.

Case 1: $\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}<0.5$. In this case, note that $\left(\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee 0.5\right)\geq 0.5$. Then, we have

$$\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\left(\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee 0.5\right)=\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}.$$

Case 2: $\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}>0.5$. Then, we have

$$\begin{aligned}&\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\left(\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee 0.5\right)\\&=\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\\&=\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}.\end{aligned}$$

Case 3: Otherwise. It means $\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\geq 0.5$ and $\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\leq 0.5$. Then, we have

$$\begin{aligned}&\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\left(\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee 0.5\right)\\&=\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge 0.5=0.5.\end{aligned}$$

The equality (4) is proved. Note that

$$\begin{aligned}&\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\left(\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee 0.5\right)\\&=\left(\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\right)\vee\left(\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge 0.5\right)\\&=\sup_{A\in\mathcal{A},A\subset\Theta_B}\Pr\{A\}\vee\left(\inf_{A\in\mathcal{A},A\supset\Theta_B}\Pr\{A\}\wedge 0.5\right).\end{aligned}$$

Hence, the equality (5) holds. □

Lemma 2. *Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space, and let* $\Theta\in\mathcal{L}\times\mathcal{A}$ *be an event. Denote that* $\Theta_B=\{\omega\in\Omega\mid\mathcal{M}\{\gamma\in\Gamma|(\gamma,\omega)\in\Theta\}\in B\}$ *for any Borel set B. Then, we have*

$$\Pr\{\Theta_B\}+\Pr\{\Theta_{B^c}\}=1$$

Proof. According to the equivalent definition of $\Pr\{\cdot\}$ in Lemma 1, we have

$$\begin{aligned}
\Pr\{\Theta_{B^c}\} &= \inf_{A\in\mathcal{A},A\supset\Theta_{B^c}} \Pr\{A\} \wedge \left(\sup_{A\in\mathcal{A},A\subset\Theta_{B^c}} \Pr\{A\} \vee 0.5\right) \\
&= \inf_{A\in\mathcal{A},A^c\subset\Theta_B} \Pr\{A\} \wedge \left(\sup_{A\in\mathcal{A},A^c\subset\Theta_B} \Pr\{A\} \vee 0.5\right) \\
&= \inf_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A^c\} \wedge \left(\sup_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A^c\} \vee 0.5\right) \\
&= \inf_{A\in\mathcal{A},A\subset\Theta_B} (1-\Pr\{A\}) \wedge \left(\sup_{A\in\mathcal{A},A\subset\Theta_B} (1-\Pr\{A\}) \vee 0.5\right) \\
&= \left(1-\sup_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A\}\right) \wedge \left(\left(1-\inf_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A\}\right) \vee 0.5\right) \\
&= \left(1-\sup_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A\}\right) \wedge \left(1-\inf_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A\} \wedge 0.5\right) \\
&= 1-\sup_{A\in\mathcal{A},A\subset\Theta_B} \Pr\{A\} \vee \left(\inf_{A\in\mathcal{A},A\supset\Theta_B} \Pr\{A\} \wedge 0.5\right) \\
&= 1-\Pr\{\Theta_B\}
\end{aligned}$$

The lemma is proved. □

Lemma 3. *Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space, and let* $\Theta_1,\Theta_2\in\mathcal{L}\times\mathcal{A}$ *be two events satisfying* $\Theta_1\subset\Theta_2$*. Then, we have*

$$\Pr\{\Theta_1\}\le\Pr\{\Theta_2\}. \tag{6}$$

Proof. $\Theta_1\subset\Theta_2$, we have

$$\inf_{A\in\mathcal{A},A\supset\Theta_1} \Pr\{A\} \le \inf_{A\in\mathcal{A},A\supset\Theta_2} \Pr\{A\},$$

$$\sup_{A\in\mathcal{A},A\subset\Theta_1} \Pr\{A\} \le \sup_{A\in\mathcal{A},A\subset\Theta_2} \Pr\{A\}.$$

According to Lemma 1, we have

$$\begin{aligned}
\Pr\{\Theta_1\} &= \inf_{A\in\mathcal{A},A\supset\Theta_1} \Pr\{A\} \wedge \left(\sup_{A\in\mathcal{A},A\subset\Theta_1} \Pr\{A\} \vee 0.5\right) \\
&\le \inf_{A\in\mathcal{A},A\supset\Theta_2} \Pr\{A\} \wedge \left(\sup_{A\in\mathcal{A},A\subset\Theta_2} \Pr\{A\} \vee 0.5\right) = \Pr\{\Theta_2\}.
\end{aligned}$$

The lemma is proved. □

Theorem 3. *(Subadditivity Theorem) The chance measure is subadditive. That is, for any countable sequence of events* $\Theta_1,\Theta_2,\cdots$*, we have*

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\} \le \sum_{i=1}^{\infty}\mathrm{Ch}\left\{\Theta_i\right\}.$$

Proof. For each ω, it follows from the subadditivity of uncertain measure that

$$\mathcal{M}\left\{\gamma \in \Gamma | (\gamma,\omega) \in \bigcup_{i=1}^{\infty}\Theta_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\}.$$

Thus, for any real number r, we have

$$\left\{\omega \in \Omega | \mathcal{M}\left\{\gamma \in \Gamma | (\gamma,\omega) \in \bigcup_{i=1}^{\infty}\Theta_i\right\} \ge r\right\} \subset \left\{\omega \in \Omega | \sum_{i=1}^{\infty}\mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\right\}$$

According to Lemma 3, we have

$$\Pr\left\{\omega \in \Omega | \mathcal{M}\left\{\gamma \in \Gamma | (\gamma,\omega) \in \bigcup_{i=1}^{\infty}\Theta_i\right\} \ge r\right\} \le \Pr\left\{\omega \in \Omega | \sum_{i=1}^{\infty}\mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\right\}$$

By the definition of chance measure, we get

$$\begin{aligned}
\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\} &= \int_0^1 \Pr\left\{\omega \in \Omega | \mathcal{M}\left\{\gamma \in \Gamma | (\gamma,\omega) \in \bigcup_{i=1}^{\infty}\Theta_i\right\} \ge r\right\} \mathrm{d}r \\
&\le \int_0^1 \Pr\left\{\omega \in \Omega | \sum_{i=1}^{\infty}\mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\right\} \mathrm{d}r \\
&\le \int_0^{+\infty} \Pr\left\{\omega \in \Omega | \sum_{i=1}^{\infty}\mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\right\} \mathrm{d}r \\
&= \sum_{i=1}^{\infty}\int_0^{+\infty} \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\}\,\mathrm{d}r \\
&= \sum_{i=1}^{\infty}\int_0^{1} \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta_i\} \ge r\}\,\mathrm{d}r \\
&= \sum_{i=1}^{\infty}\mathrm{Ch}\{\Theta_i\}.
\end{aligned}$$

That is, the chance measure is subadditive. □

Null-additivity is a direct deduction from the above theorem. In fact, a more general theorem can be proved as follows.

Theorem 4. *Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space and* $\Theta_1, \Theta_2, \cdots$ *be a sequence of events with* $\mathrm{Ch}\{\Theta_i\} \to 0$ *as* $i \to \infty$. *Then, for any event* Θ, *we have*

$$\lim_{i\to\infty}\mathrm{Ch}\{\Theta \cup \Theta_i\} = \lim_{i\to\infty}\mathrm{Ch}\{\Theta \setminus \Theta_i\} = \mathrm{Ch}\{\Theta\}.$$

Proof. By using the monotonicity and subadditivity of chance measure, we have

$$\mathrm{Ch}\{\Theta\} \le \mathrm{Ch}\{\Theta \cup \Theta_i\} \le \mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta_i\} \tag{7}$$

for each i. For $\mathrm{Ch}\{\Theta_i\} \to 0$ as $i \to \infty$, we get $\mathrm{Ch}\{\Theta \cup \Theta_i\} \to \mathrm{Ch}\{\Theta\}$. Note that $\Theta \setminus \Theta_i \subset \Theta \subset ((\Theta \setminus \Theta_i) \cup \Theta_i)$. We have

$$\mathrm{Ch}\{\Theta \setminus \Theta_i\} \le \mathrm{Ch}\{\Theta\} \le \mathrm{Ch}\{\Theta \setminus \Theta_i\} + \mathrm{Ch}\{\Theta_i\}. \tag{8}$$

Hence, $\lim_{i\to\infty}\mathrm{Ch}\{\Theta \setminus \Theta_i\} = \mathrm{Ch}\{\Theta\}$. □

Remark. From the above theorem, we know that the chance measure is null-additive. That means $\mathrm{Ch}\{\Theta_1 \cup \Theta_2\} = \mathrm{Ch}\{\Theta_1\} + \mathrm{Ch}\{\Theta_2\}$ if either $\mathrm{Ch}\{\Theta_1\} = 0$ or $\mathrm{Ch}\{\Theta_2\} = 0$. □

Theorem 5. *(Asymptotic Theorem) Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space. For any events* $\Theta_1, \Theta_2, \cdots$, *we have*

$$\lim_{i\to\infty} \mathrm{Ch}\{\Theta_i\} > 0, \quad \text{if } \Theta_i \uparrow \Gamma \times \Omega, \tag{9}$$

$$\lim_{i\to\infty} \mathrm{Ch}\{\Theta_i\} < 1, \quad \text{if } \Theta_i \downarrow \emptyset. \tag{10}$$

Proof. Assume $\Theta_i \uparrow \Gamma \times \Omega$. Since $\Gamma \times \Omega = \cup_i \Theta_i$, it follows from the subadditivity of chance measure that

$$1 = \mathrm{Ch}\{\Gamma \times \Omega\} \le \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}.$$

Note that $\mathrm{Ch}\{\Theta_i\}$ is increasing with respect to i. We get $\lim_{i\to\infty} \mathrm{Ch}\{\Theta_i\} > 0$. If $\Theta_i \downarrow \emptyset$, then $\Theta_i^c \uparrow \Gamma \times \Omega$. By using inequality (9) and the duality of chance measure, we have

$$\lim_{i\to\infty} \mathrm{Ch}\{\Theta_i\} = 1 - \lim_{i\to\infty} \mathrm{Ch}\{\Theta_i^c\} < 1.$$

The theorem is proved. □

Competing interests
This paper proposed several properties of chance space. Besides, the subadditivity theorem, null-additivity theorem, and asymptotic theorem of chance measure were proved.

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