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Existence and uniqueness of the solution to uncertain fractional differential equation

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Abstract
The paper proves an existence and uniqueness theorem of the solution to an uncertain fractional differential equation by Banach fixed point theorem under Lipschitz and linear growth conditions. Then, the paper presents an existence theorem for the solution of an uncertain fractional differential equation by Schauder fixed point theorem under continuity condition.

Keywords: Uncertainty theory; Fractional derivative; Uncertain fractional differential equation; Existence; Uniqueness

Introduction

Fractional differential equation has been a classical research field of differential equations. The study of fractional differential equations attracted many mathematicians, physicists, and engineers. Fractional differential equations have important applications such as in rheology, viscoelasticity, electrochemistry, and electromagnetism. A fractional differential equation is a differential equation including fractional derivatives. There are several kinds of fractional derivatives such as the Riemann-Liouville type, Caputo type, Grünwald-Letnikov type, and Riesz type. Some references about fractional differential equations may be seen in [1-8].

Stochastic fractional differential equations were used to model dynamical systems affected by random noises [9-15]. Generally, there are two types of stochastic fractional differential equations. One is of the form

$$D^{\alpha}x(t) = f(t, x(t)) + g(t, x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}$$

where $D^{\alpha}x(t)$ denotes the fractional derivative of the function $x(t)$, and $W(t)$ is the Wiener process. The other is of the form

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}B_t^H$$

where B_t^H is the fractional Brownian motion.

Recently in [16], the concept of uncertain fractional differential equations was introduced based on the uncertainty theory. The Riemann-Liouville type of uncertain fractional differential equation

$$D^{p}X_t = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}$$

and the Caputo type of uncertain fractional differential equation

$$^{c}D^{p}X_t = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}$$

were dealt with where C_t is the canonical Liu process. The solutions were provided by the Mittag-Leffler function for linear uncertain fractional differential equations. An application to the price of a zero-coupon bond with a maturity date was presented.

Analytical solutions were presented only for some special uncertain fractional differential equations in [16]. In order to understand what kinds of uncertain fractional differential equations have solutions, we in this paper will give some sufficient conditions to guarantee the existence of solutions of uncertain fractional differential equations. That is, we will show the existence and uniqueness of solutions for uncertain fractional differential equations. The structure of the paper is as follows: Firstly, some concepts and results in uncertainty theory will be reviewed. Then, the fractional derivatives and uncertain fractional differential equations will be recalled. Finally, an existence and uniqueness theorem and an existence theorem will be proved.

Preliminary

Uncertainty theory was founded by Liu in 2007 [17] and refined in 2010 [18]. Basic concepts in uncertainty theory include uncertain measure, uncertainty space, product uncertain measure, and uncertain variable. Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. Set function $\mathcal{M}$ defined on $\mathcal{L}$ is called an uncertain measure if it satisfies three axioms: (normality) $\mathcal{M}\{\Gamma\} = 1$, (duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ, and (countable subadditivity) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying $\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$, where $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ are uncertainty spaces and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively. An uncertain variable is defined as a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set R of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B.

The uncertainty distribution $\Phi : R \to [0, 1]$ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for $x \in R$. The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. The variance of ξ is defined by $V[\xi] = E\left[(\xi - E[\xi])^2\right]$.

A normal uncertain variable with expected value e and variance σ^2 has the uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1},\ x \in R$$

which is denoted by $\xi \sim \mathcal{N}(e, \sigma)$.

The uncertain variables $\xi_1, \xi_2, \cdots \xi_m$ are said to be independent [19] if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots B_m$ of real numbers. For numbers a and b, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ if ξ and η are independent uncertain variables.

Liu [20] defined uncertain process as a function X_t from $S \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers where S is a totally ordered set such that $\{X_t \in B\}$ is an event for any Borel set B at each time $t \in S$.

Definition 1. *[19] An uncertain process C_t is called a canonical Liu process if it satisfies the following: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, and (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, denoted by $C_{s+t} - C_s \sim \mathcal{N}(0, t)$.*

For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|$. Then, the uncertain integral of X_t with respect to C_t is defined by Liu [19] as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot \left(C_{t_{i+1}} - C_{t_i}\right)$$

provided that the limit exists almost surely and is finite. If there exist two uncertain processes μ_t and σ_t such that $Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$ for any $t \ge 0$, then we say Z_t has an uncertain differential $\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t$. An uncertain differential equation driven by a canonical Liu process C_t is defined as

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

where f and g are two given functions. A solution X_t of the uncertain differential equation is equivalent to a solution of the uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s.$$

For an uncertain differential equation (1) in a multidimensional case, X_t is a multidimensional state, C_t is a multidimensional canonical Liu process, $f(t, X_t)$ is a vector-valued function, and $g(t, X_t)$ is a matrix-valued function.

An existence and uniqueness theorem of solution for the uncertain differential equation (1) was proved by Chen and Liu [21]. Meanwhile, Chen and Liu [21] obtained an analytic solution to linear uncertain differential equations. Liu [22] and Yao [23] presented some methods for solving nonlinear uncertain differential equations. Yao and Chen [24] introduced a numerical method for solving the uncertain differential equation. Some extensions of the uncertain differential equation were studied such as the uncertain delay differential equation by Barbacioru [25], Ge and Zhu [26], and Liu and Fei [27] and the backward uncertain differential equation by Ge and Zhu [28]. The uncertain differential equation has been applied in some fields such as uncertain finance [29], uncertain optimal control [30], and uncertain differential game [31].

Lemma 1. *[28] Suppose that C_t is an l-dimensional canonical Liu process, and Y_t is an integrable $n \times l$-dimensional uncertain process on $[a, b]$ with respect to t. Then, the inequality*

$$\left\| \int_a^b Y_t(\gamma) dC_t(\gamma) \right\|_\infty \le K_\gamma \int_a^b \|Y_t(\gamma)\|_\infty dt$$

holds, where K_γ is the Lipschitz constant of the sample path $C_t(\gamma)$ with the norm $\| \cdot \|_\infty$.

Uncertain fractional differential equations

In the sequel, we will always assume $p \in (0, 1]$. The Riemann-Liouville type of uncertain fractional differential equation and the Caputo type of uncertain fractional differential equation in the one-dimensional case were introduced in [16]. Now we state those concepts in a multidimensional case. Let $C_t = (C_{1t}, C_{2t}, \cdots, C_{lt})^\tau$ be an l-dimensional canonical Liu process.

Definition 2. *Suppose that $f : [0, \infty) \times R^n \to R^n$ and $g : [0, \infty) \times R^n \to R^{n \times l}$ are two functions. Then,*

$$D^p X_t = f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \tag{2}$$

is called an uncertain fractional differential equation of the Riemann-Liouville type. A solution of (2) with the initial condition

$$\lim_{t \to 0+} t^{1-p} X_t = x_0$$

is an uncertain process X_t such that

$$X_t = t^{p-1} x_0 + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_s) \mathrm{d}s + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_s) \mathrm{d}C_s \tag{3}$$

holds almost surely.

Definition 3. *Suppose that $f : [0, \infty) \times R^n \to R^n$ and $g : [0, \infty) \times R^n \to R^{n \times l}$ are two functions. Then,*

$${}^c D^p X_t = f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \tag{4}$$

is called an uncertain fractional differential equation of the Caputo type. A solution of (4) is an uncertain process X_t such that

$$X_t = X_0 + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_s) \mathrm{d}s + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_s) \mathrm{d}C_s \tag{5}$$

holds almost surely.

Remark 1. (i) The pth Riemann-Liouville fractional order derivative of the function $u : [0, T] \to R^n$ is defined by

$$D^p u(t) = \frac{1}{\Gamma(1-p)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-p} u(s) \mathrm{d}s, \quad t > 0.$$

(ii) The pth Caputo fractional order derivative of the function $u : [0,T] \to R^n$ is defined by

$$ {}^cD^p u(t) = \frac{1}{\Gamma(1-p)} \int_0^t (t-s)^{-p} u'(s) \mathrm{d}s, \quad t > 0 $$

where $u'(s)$ is the first-order derivative of $u(s)$.
(iii) The relation between the Riemann-Liouville and Caputo fractional order derivatives is

$$ D^p u(t) = {}^cD^p u(t) + \frac{t^{-p}}{\Gamma(1-p)} u(0). $$

(iv) The gamma function

$$ \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t, \quad \alpha > 0 $$

has the properties

$$ \Gamma(\alpha+1) = \alpha\Gamma(\alpha),\ \alpha > 0;\ \Gamma(1) = 1;\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. $$

Existence and uniqueness

For simplicity, we use $|\cdot|$ to denote a norm in R^n or $R^{n\times l}$. Let $D_{[a,b]}$ denote the space of continuous R^n-valued functions on $[a,b]$, which is a Banach space with the norm

$$ \|x_t\| = \max_{t\in[a,b]} |x_t|, \quad \text{for } x_t \in D_{[a,b]}. $$

Give two functions $f(t,x) : [0,T] \times R^n \to R^n$, and $g(t,x) : [0,T] \times R^n \to R^{n\times l}$. Now we introduce the following mapping Φ on $D_{[0,T]}$: for $X_t \in D_{[0,T]}$,

$$ \Phi(X_t) = t^{p-1}x_0 + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s,X_s) \mathrm{d}s + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s,X_s) \mathrm{d}C_s \quad (6) $$

where x_0 is a given initial state.

Lemma 2. *For uncertain process $X_t \in D_{[0,T]}$, the mapping Ψ defined by*

$$ \begin{aligned} \Psi(X_t) = & \left(t^{p-1} - \tilde{a}\right) x_0 + X_a + \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s,X_s) \mathrm{d}s \\ & + \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} g(s,X_s) \mathrm{d}C_s, \quad t > a \ge 0 \end{aligned} \quad (7) $$

is sample-continuous where $\tilde{a} = a^{p-1}$ if $a > 0$ or 1 if $a = 0$, and f and g satisfy the linear growth condition

$$ |f(t,x)| + |g(t,x)| \le L(1+|x|), \quad \forall x \in R^n, \quad t \in [0,+\infty) $$

where L is a positive constant.

Proof. In fact, for $\gamma \in \Gamma$ and $t > r > a$, we have

$$
\begin{aligned}
&|\Psi(X_t(\gamma)) - \Psi(X_r(\gamma))| \\
&= \left| \left(t^{p-1} - r^{p-1}\right) x_0 + \frac{1}{\Gamma(p)} \int_r^t (t-s)^{p-1} f(s, X_s(\gamma)) \mathrm{d}s \right. \\
&\quad + \frac{1}{\Gamma(p)} \int_r^t (t-s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \\
&\quad + \frac{1}{\Gamma(p)} \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] f(s, X_s(\gamma)) \mathrm{d}s \\
&\quad \left. + \frac{1}{\Gamma(p)} \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \right| \\
&\le \left(r^{p-1} - t^{p-1}\right) |x_0| + \frac{1}{\Gamma(p)} \int_r^t (t-s)^{p-1} |f(s, X_s(\gamma))| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(p)} \left| \int_r^t (t-s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \right| \\
&\quad + \frac{1}{\Gamma(p)} \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] |f(s, X_s(\gamma))| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(p)} \left| \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \right| \\
&\le \left(r^{p-1} - t^{p-1}\right) |x_0| + \frac{1}{\Gamma(p)} \int_r^t (t-s)^{p-1} |f(s, X_s(\gamma))| \mathrm{d}s \\
&\quad + \frac{K_\gamma}{\Gamma(p)} \int_r^t (t-s)^{p-1} |g(s, X_s(\gamma))| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(p)} \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] |f(s, X_s(\gamma))| \mathrm{d}s \\
&\quad + \frac{K_\gamma}{\Gamma(p)} \int_a^r \left[(t-s)^{p-1} - (r-s)^{p-1}\right] \left|g(s, X_s(\gamma))\right| \mathrm{d}s \quad \text{(by Lemma 1)} \\
&\le \left(r^{p-1} - t^{p-1}\right) |x_0| + \frac{L}{\Gamma(p+1)} (1 + \|X_t(\gamma)\|)(1 + K_\gamma) \left[(t-a)^p - (r-a)^p\right]
\end{aligned}
$$

by the linear growth condition. Thus, $|\Psi(X_t(\gamma)) - \Psi(X_r(\gamma))| \to 0$ as $|t-r| \to 0$. That is, $\Psi(X_t)$ is sample-continuous. □

Theorem 1. *(Existence and uniqueness) The uncertain fractional differential equation (2) (or (4)) has a unique solution X_t in $[0, +\infty)$ if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the Lipschitz condition*

$$
|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le L|x-y|, \quad \forall x, y \in R^n, \quad t \in [0, +\infty)
$$

and the linear growth condition

$$
|f(t,x)| + |g(t,x)| \le L(1 + |x|), \quad \forall x \in R^n, \quad t \in [0, +\infty)
$$

where L is a positive constant. Furthermore, X_t is sample-continuous.

Proof. We only prove the theorem for the uncertain fractional differential equation (2). A similar process of the proof is suitable to the uncertain fractional differential equation (4). Let $T > 0$ be an arbitrarily given number, and let Φ be a mapping defined by (6) on $D_{[0,T]}$.

Give $\gamma \in \Gamma$. For $\lambda \in [0, T)$, assume $c > 0$ such that $\lambda + c \le T$. Define a mapping ψ on $D_{[\lambda,\lambda+c]}$: for $X_t \in D_{[\lambda,\lambda+c]}$, $t \in [\lambda, \lambda + c]$,

$$\psi(X_t) = \left(t^{p-1} - \tilde{\lambda}\right) x_0 + X_\lambda + \frac{1}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} f(s, X_s) \mathrm{d}s + \frac{1}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} g(s, X_s) \mathrm{d}C_s$$

where $\tilde{\lambda} = \lambda^{p-1}$ if $\lambda > 0$ or 1 if $\lambda = 0$. For $X_t(\gamma) \in D_{[\lambda,\lambda+c]}$, we know that $\psi(X_t(\gamma)) \in D_{[\lambda,\lambda+c]}$ by Lemma 2.

Let $X_t(\gamma), Y_t(\gamma) \in D_{[\lambda,\lambda+c]}$. For any $t \in [\lambda, \lambda + c]$, we have

$$\begin{aligned}
&\|\psi(X_t(\gamma)) - \psi(Y_t(\gamma))\| = \max_{t\in[\lambda,\lambda+c]} |\psi(X_t(\gamma)) - \psi(Y_t(\gamma))| \\
&\le \max_{t\in[\lambda,\lambda+c]} \left| \frac{1}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} \left[f(s, X_s(\gamma)) - f(s, Y_s(\gamma))\right] \mathrm{d}s \right. \\
&\quad \left. + \frac{1}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} \left[g(s, X_s(\gamma)) - g(s, Y_s(\gamma))\right] \mathrm{d}C_s(\gamma) \right| \\
&\le \max_{t\in[\lambda,\lambda+c]} \left\{ \frac{1}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} |f(s, X_s(\gamma)) - f(s, Y_s(\gamma))| \mathrm{d}s \right. \\
&\quad \left. + \frac{K_\gamma}{\Gamma(p)} \int_\lambda^t (t-s)^{p-1} |g(s, X_s(\gamma)) - g(s, Y_s(\gamma))| \mathrm{d}s \right\} \\
&\le \frac{(1+K_\gamma)L}{\Gamma(p)} \max_{t\in[\lambda,\lambda+c]} \int_\lambda^t (t-s)^{p-1} |X_s(\gamma) - Y_s(\gamma)| \mathrm{d}s \quad \text{(by Lipschitz condition)} \\
&\le \frac{(1+K_\gamma)Lc^p}{\Gamma(p+1)} \|X_t(\gamma) - Y_t(\gamma)\|.
\end{aligned}$$

Let $\rho(\gamma) = \left(1 + K_\gamma\right) Lc^p / \Gamma(p+1)$. By taking a suitable $c = c(\gamma) > 0$ such that $\rho(\gamma) \in (0, 1)$. That is, ψ is a contraction mapping on $D_{[\lambda,\lambda+c]}$. Thus, by the well-known Banach fixed point theorem, we have a unique fixed point $X_t(\gamma)$ of ψ in $D_{[\lambda,\lambda+c]}$. Furthermore, $X_t(\gamma) = \lim_{k\to\infty} \psi(X_{t,k}(\gamma))$ where

$$X_{t,k}(\gamma) = \psi(X_{t,k-1}(\gamma)), \ k = 1, 2, \cdots$$

for any given $X_{t,0}(\gamma) = x_t \in D_{[\lambda,\lambda+c]}$.

Assume that $[0, c], [c, 2c], \cdots, [kc, T]$ are the subsets of $[0, T]$ with $kc < T \le (k + 1)c$. The above process implies that the mapping ψ has a unique fixed point $X_t^{(i+1)}(\gamma)$ with $X_{ic}^{(i+1)}(\gamma) = X_{ic}^{(i)}(\gamma)$ on the interval $[ic, (i + 1)c]$ for $i = 0, 1, 2, \cdots, k$, where we set $(k + 1)c = T$. Define $X_t(\gamma)$ on the interval $[0, T]$ by setting

$$X_t(\gamma) = \begin{cases} X_t^{(1)}(\gamma), & t \in [0, c], \\ X_t^{(2)}(\gamma), & t \in [c, 2c], \\ \cdots \\ X_t^{(k+1)}(\gamma), & t \in [kc, T]. \end{cases}$$

It is easy to see that $X_t(\gamma)$ is the unique fixed point of Φ defined by (6) in $D_{[0,T]}$. In addition, $X_t(\gamma) = \lim_{k\to\infty} \Phi(X_{t,k}(\gamma))$ where

$$X_{t,k}(\gamma) = \Phi(X_{t,k-1}(\gamma)),\ k = 1, 2, \cdots$$

for any given $X_{t,0}(\gamma) = x_t \in D_{[0,T]}$. Since $X_{t,k}$ are uncertain vectors for $k = 1, 2, \cdots$, we know that X_t is an uncertain vector by Theorem 3 in the Appendix. It follows from the arbitrariness of $T > 0$ that X_t is the unique solution of uncertain fractional differential equation (2). Furthermore, since $X_t(\gamma)$ is in $D_{[0,T]}$, X_t is sample-continuous. The theorem is proved. □

If the functions f and g do not satisfy the Lipschitz condition and linear growth condition, we present the following existence theorem just for continuous f and g.

Theorem 2. *(Existence) Let $f(t, x)$ and $g(t, x)$ be continuous in*

$$G = [0, T] \times \left\{x \in R^n : |x - x_0| \le b\right\}.$$

Then, uncertain fractional differential equation of the Caputo type (4) has a solution X_t in $t \in [0, T]$ with the crisp initial condition $X_0 = x_0 \in R^n$.

Proof. For any $\gamma \in \Gamma$, let $c > 0$ be a positive number such that

$$\frac{M\left(1 + K_\gamma\right)}{\Gamma(p+1)} c^p = b$$

where K_γ is the Lipschitz constant of the canonical Liu process C_t, and $M = \max_{(t,x)\in G} |f(t, x)| \vee |g(t, x)|$. Denote

$$H = \left\{X_t(\gamma) \in D_{[0,h]} : X_t \text{ is an uncertain vector and } \|X_t(\gamma) - x_0\| \le \frac{M(1 + K_\gamma)}{\Gamma(p+1)} h^p\right\}$$

where $h = \min\{T, c\}$.

It is easy to see that H is a closed convex set. Define a mapping Φ on H by

$$\begin{aligned}\Phi(X_t(\gamma)) = x_0 &+ \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} f(s, X_s(\gamma)) \mathrm{d}s \\ &+ \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma), \quad 0 \le t \le h.\end{aligned} \tag{8}$$

For $X_t(\gamma) \in H$, we have

$$\begin{aligned}\|\Phi(X_t(\gamma)) - x_0\| &= \max_{0\le t\le h} |\Phi(X_t(\gamma)) - x_0| \\ &\le \max_{0\le t\le h} \left\{\frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} |f(s, X_s(\gamma))| \mathrm{d}s \right. \\ &\quad \left. + \frac{K_\gamma}{\Gamma(p)} \int_0^t (t - s)^{p-1} |g(s, X_s(\gamma))| \mathrm{d}s\right\} \\ &\le \max_{0\le t\le h} \frac{M(1 + K_\gamma)}{\Gamma(p)} \int_0^t (t - s)^{p-1} \mathrm{d}s \\ &\le \frac{M(1 + K_\gamma)}{\Gamma(p+1)} h^p.\end{aligned} \tag{9}$$

That is $\Phi(X_t(\gamma)) \in H$, and the mapping Φ is bounded uniformly in $X_t(\gamma) \in H$. In addition, for $0 \le t_1 < t_2 \le h$, it is easy to verify

$$\begin{aligned}
|\Phi(X_{t_1}(\gamma)) - \Phi(X_{t_2}(\gamma))| = & \left| \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} f(s, X_s(\gamma)) \mathrm{d}s \right. \\
& + \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \\
& + \frac{1}{\Gamma(p)} \int_0^{t_1} \left[(t_2 - s)^{p-1} - (t_1 - s)^{p-1} \right] f(s, X_s(\gamma)) \mathrm{d}s \\
& \left. + \frac{1}{\Gamma(p)} \int_0^{t_1} \left[(t_2 - s)^{p-1} - (t_1 - s)^{p-1} \right] g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \right| \\
\le & \frac{M\left(1 + K_\gamma\right)}{\Gamma(p+1)} \left(t_2^p - t_1^p\right)
\end{aligned}$$

which comes to a conclusion that Φ is equicontinuous for $X_t(\gamma) \in H$ in $[0, h]$. It follows from the Ascoli-Arzela theorem that Φ is a compact mapping on H.

Let $X_{t,i}(\gamma)$ converge to $X_t(\gamma)$ in H as $i \to \infty$. That is, $X_{t,i}(\gamma)$ converges to $X_t(\gamma)$ uniformly in $t \in [0, h]$. Thus,

$$\begin{aligned}
\Phi(X_{t,i}(\gamma)) = & \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} f(s, X_{s,i}(\gamma)) \mathrm{d}s \\
& + \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} g(s, X_{s,i}(\gamma)) \mathrm{d}C_s(\gamma) \\
\to & \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} f(s, X_s(\gamma)) \mathrm{d}s \\
& + \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma) \\
= & \Phi(X_t(\gamma))
\end{aligned}$$

uniformly in $t \in [0, h]$. This shows that Φ is continuous on H.

It follows from the Schauder fixed point theorem that Φ has a fixed point $X_t(\gamma)$ on H. Hence,

$$\begin{aligned}
X_t(\gamma) = x_0 & + \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} f(s, X_s(\gamma)) \mathrm{d}s \\
& + \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} g(s, X_s(\gamma)) \mathrm{d}C_s(\gamma)
\end{aligned} \tag{10}$$

for $t \in [0, h]$. By the extension method, there exists $X_t(\gamma)$ satisfying (10) in $t \in [0, T]$. That is, X_t is a solution of (4). Therefore, the conclusion of the theorem is proved. □

Conclusions

For uncertain fractional differential equations of the Riemann-Liouville type and Caputo type, an existence and uniqueness theorem of solution was presented by employing the Banach fixed point theorem under sufficient conditions that the drift and diffusion functions are linear growing and Lipschitzian. If the drift and diffusion functions are just continuous, an existence theorem of solution for the uncertain fractional differential equation of the Caputo type was proved by employing the Schauder fixed point theorem. The existence and uniqueness of solution to the uncertain fractional differential equation will give a theoretical foundation for studying the stability of uncertain fractional differential equations and uncertain finance.

Appendix

Theorem 3. *Let $\xi_1, \xi_2, \cdots$ be uncertain variables and $\lim_{i\to\infty} \xi_i = \xi$ almost surely. Then, ξ is an uncertain variable.*

Proof. Let

$$\mathcal{B} = \{B \subset R : \{\xi \in B\} \text{is an event}\}.$$

Then, $\mathcal{B}$ is a σ-algebra. For $a, b \in R$, since

$$\xi = \sup_{n\geq 1} \inf_{i\geq n} \xi_i = \inf_{n\geq 1} \sup_{i\geq n} \xi_i,$$

we have

$$\{\xi < b\} = \left\{ \sup_{n\geq 1} \inf_{i\geq n} \xi_i < b \right\} = \bigcup_{k\geq 1} \bigcap_{n\geq 1} \bigcup_{i\geq n} \{\xi_i \leq b - \epsilon_k\},$$

$$\{\xi > a\} = \left\{ \inf_{n\geq 1} \sup_{i\geq n} \xi_i > a \right\} = \bigcup_{k\geq 1} \bigcap_{n\geq 1} \bigcup_{i\geq n} \{\xi_i \geq a + \epsilon_k\}$$

where $\{\epsilon_k\}$ is a sequence of positive numbers converging decreasingly to zero. Since ξ_i are uncertain variables for all i, we know that $\{\xi_i \leq b-\epsilon_k\}$ and $\{\xi_i \geq a+\epsilon_k\}$ are events. Hence, $\{\xi < b\}$ and $\{\xi > a\}$ are events and then $\{a < \xi < b\}$ is an event. That is, $(a, b) \in \mathcal{B}$. Since the smallest σ-algebra containing all open intervals of R is just Borel algebra over R, the class $\mathcal{B}$ contains all Borel sets. That is, for any Borel set B, $\{\xi \in B\}$ is an event. Therefore, ξ is an uncertain variable. □

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