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Controller Parameter Optimization for the Robust Synchronization of Chaotic Systems with Known and Unknown Parameters

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Abstract

In this paper, a synchronization problem of a three-dimensional (3-D) Coullete chaotic system using the active- and adaptive-based synchronization control techniques is addressed. Based on the Routh-Hurwitz criterion and using the active control algorithm, a single control function is considered and a computational study is performed to identify the correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the linear controlling parameters (LCPs) for the globally exponential synchronization (GES) between two identical 3-D Coullete chaotic systems. In order to achieve the complete synchronization (CS) objective with unknown model uncertainties, external disturbances, and unknown time-varying parameters, a novel nonlinear adaptive synchronous controller is proposed and suitable adaptive laws of time-varying parameters are designed that accomplish the asymptotic synchronization between two identical uncertain 3-D Coullete chaotic systems. The two synchronizing controlling approaches are applied to investigate the CS phenomenon, and the results are compared. Open research problems are also discussed. All simulations results are carried out to validate the effectiveness of the proposed synchronization control approaches by using *Mathematica 10.0*.

Keywords: Chaos synchronization, Active control, Adaptive control, Routh-Hurwitz criterion, Lyapunov stability theory

Introduction

Chaos synchronization can be considered as the simplest types of cooperation between chaotic systems. This cooperation can be induced by coupling or an external force. Through a weak coupling of the chaotic systems, the difference in behaviors of the two coupled chaotic systems under different initial conditions goes to zero when time tends to infinity. This idea of chaos synchronization was first introduced in [1]. After the remarkable work [1], chaos synchronization has been widely investigated in the relevant literature [2–4]. At present, chaos synchronization has received increasing interest in many scientific disciplines, such as chemical processes and biological systems [5], cryptosystem [6], information processing [7], secure communications [8], and many physical systems [9]. As a result, a variety of control methods and techniques have been developed to study the chaos synchronization. These include the adaptive control strategy [9], projective synchronization [10], sliding mode control [11], nonlinear control techniques [12], and active control method [13], which are well-known synchronization control techniques among others. Among these techniques, the active control (AC) and adaptive synchronization control (ASC) strategies have attracted a great interest in the literature concerned because of their easy implementation to practical systems [14–22]. The AC method for the chaos synchronization was first proposed in [17] based on the active control theory [23] and further studied by many researchers [24–26] among others. The AC strategy can be easily designed according to the given conditions of coupled chaotic (or hyperchaotic) systems to achieve the chaotic synchronization globally exponentially. The AC technique for chaos synchronization is used when the nonlinearity and parameters of the coupled chaotic systems are known [17] (Additional file 1).

In practical applications, some or all of the system's parameters are not known in advance. These parameters change from time to time [27]. Therefore, to tackle the analytical and computational stability complications produced by the parameter uncertainties, the ASC approach is used. The ASC strategy is based on the Lyapunov stability theory [28] and yields the asymptotic tracking of the closed-loop system with all remaining signals bounded in the presence of the system uncertainties [27, 28].

Past studies of the chaos synchronization using AC strategy [13–17, 24–26] have more concentrated on the fast convergence rates of the synchronization error signals. For this purpose, the closed-loop stability for the CS is established by using huge control functions. This demands an extra effort on the controller design. Furthermore, it creates two important issues in the synchronization process [27]. Firstly, these control functions are responsible for the ineffective use of energy due to the creation of large amplitude oscillation of the synchronized error signals. This may give birth to signal saturations, thus resulting in the loss of synchronization stability completely. Secondly, by just placing the poles of the linearized error system to the left half of the complex, many possible choices are available for the construction of linear controller parameters (LCPs). With this hypothesis, the message signal could be easily extracted from the communications channel during the transmission because of any possible choice of the LCPs [4]. This may lead to security problems. Moreover, there is no precise balance between the converging rates of the synchronized error signals to the origin and magnitude of the LCPs. Similarly, it has also been established from past studies [13–17, 24–26] that the feedback control inputs must be applied to all states of the error system. This places extra burdens on the controller design and complicating the AC strategy for the chaotic synchronization. Nevertheless, chaos synchronization with lower control signals and the correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the LCPs for the GES bear great importance in practical applications and have received little attention in the literature concerned.

From the literature survey, it has also been established that while using the ASC approaches [18–22] (among others), the feedback control functions for the CS of two coupled chaotic (or hyperchaotic) systems have been developed so that the amplitude of the oscillation of the error signals and the synchronization transient time are large. However, the variations in amplitude of the oscillation of the error signals cannot be evaded during the transmission of a chaotic signal, which actually determines the amount of energy to transmit the chaotic signals from transmitter to receiver. Likewise, the time response is an important feature in real-time applications such as communication networks as they depend on the time synchronization between different nodes [29]. Furthermore, in [18–22], the unknown parameters are assumed to be constant and the chaotic systems are considered free of the unknown model uncertainties and external disturbances. These perturbations are not integrated into the system dynamics altogether due to the analytical and computational stability complications. In the presence of these perturbations, the closed-loop for chaos synchronization may lose the synchronization stability completely.

In spite of various efforts in the study of CS, a considerable attention is still required, especially in five cases: (i) low synchronization controller cost, (ii) to determine a correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the LCPs, (iii) the quick response of the controller to converge the synchronization error signals to the origin as soon as possible, (iv) to suppress the amplitude of the synchronized error signals, and (v) when the uncertain parameters are time-varying instead of real constant.

In view of the aforesaid issues, the main aim of this paper is to design such feedback controller functions that will improve the performance of AC and ASC strategies for the CS of chaotic (or hyperchaotic) systems. Based on the Routh-Hurwitz criterion [23] and using the AC strategy, a single control function is proposed and a computational study has been performed to identify the correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the LCPs so that they establish the GES between two identical Coullete chaotic systems [30] under the determined parameters. Since it is obvious that the active controller is based on linearizing the error system, it is highly sensitive to any change in the parameters of the system, unknown model uncertainties, and external disturbances. Therefore, to tackle the analytical and computational stability complications produced by different types of perturbations, a novel nonlinear controller function is constructed and suitable adaptive laws of time-varying parameters are developed, respectively to accomplish the robust synchronization between two identical Coullete chaotic systems in the presence of unknown model uncertainties and external disturbances. All the time-varying parameters with different numerical values are identified accurately. The closed-loop is stabilized at the origin with faster synchronization speed.

As compared to the past published works in the relevant literature, the main contributions of this study include the following: (i) identification of the correct balance between the magnitude of the LCPs and the converging rates of the synchronized error signals to the origin, (ii) computation of the suitable position for the LCPs in the complex plane for the GES, (iii) less control effort and faster synchronization speed, (iv) low amplitude of the oscillation of the error signals, and (v) designing of a novel adaptive robust CS controller in the presence of unknown model uncertainties, external disturbances, and unknown time-varying parameters. In terms of the synchronization speed, rates, quality, and cost of the synchronization controller, a comparative study has been performed between the present study in this paper with the previous results in [16, 27, 30] to validate the performance of the proposed AC and ASC algorithm approaches.

Since the 3-D Coullete chaotic system represents a nonlinear circuit, demonstrating chaotic behavior, it is capable of synchronizing chaotic communications and suitable for transmission of digital signals with minimum synchronization error in practical applications. The proposed algorithm approaches can be successfully applied to two coupled Coullete chaotic systems for cryptosystem [6], information and image processing [7], and secure communications [8] under different circumstances, where the security of the synchronization process is of top priority.

The rest of the paper is organized as follows: In the "Complete Synchronization" section, a description of the Coullete chaotic system is presented and the problem statements for the CS with known and unknown parameters are formulated. In the "Solution" section, solutions to the synchronization problem of the two identical Coullete chaotic systems using the AC and ASC strategies are presented. Finally, the concluding remarks are given in the "Conclusions" section, with some future research works.

Complete Synchronization

Model of the Coullete Chaotic System

The heterogeneous chaotic circuits are one of the natural systems, and it can be-characterized by a variety of equations [31]. The chaotic attractors exhibited by heterogeneous systems have received considerable attention of the researchers theoretically as well as experimentally [16, 32]. The 3-D Coullete chaotic system is one of the heterogeneous systems, which realizes chaos and shows very rich and complex dynamic behavior and can be useful for secure communications [31]. The Coullete chaotic system [30] contains a single cubic term and three positive parameters. The vector form of the Coullete chaotic system [30] is given as follows:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x^3(t) \end{pmatrix} \tag{1}$$

where $[x(t),\ y(t),\ z(t)] \in R^3$ are the state variables and a, b, and c are the positive parameters of the system (1). The Coullete system (1) exhibits a chaotic attractor for the parameter values $a = 5.5$, $b = 3.5$, and $c = 1$ with initial condition $x(t) = 0.145$, $y(t) = 0.625$, and $z(t) = 0.925$, as shown in Fig. 1a, while Fig. 1b shows the time history of the state variables of the Coullete chaotic system (1).

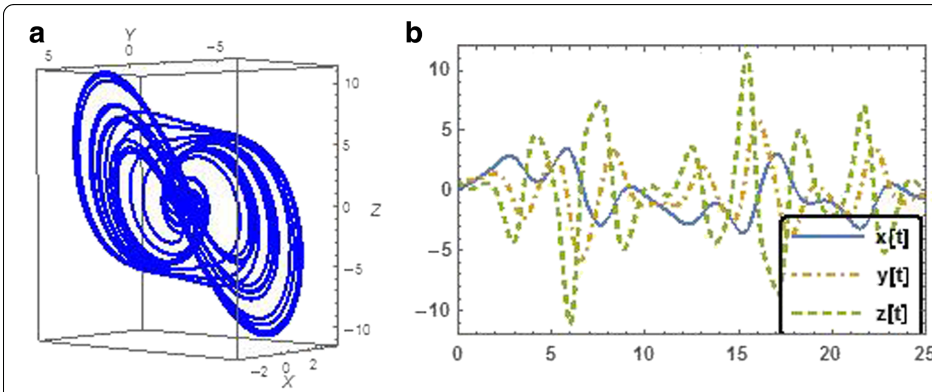

Fig. 1 **a** 3-D attractor of the Coullete chaotic system in *XYZ* space. **b** Time history of the state variables of Coullete chaotic system

Problem Statements

Synchronization Between Two Identical Coullete Chaotic Systems Using Active Control Strategy

This subsection presents the problem formulation for the CS of two identical Coullete chaotic systems (1). For this purpose, we consider two Coullete chaotic systems, where the master Coullete chaotic system with three state variables denoted by the subscript 1 drives the slave Coullete chaotic system with a feedback controller having identical equations denoted by the subscript 2. However, the initial condition of the master Coullete system is different from that of the slave Coullete system. The two coupled identical Coullete chaotic systems are described in a master-slave system synchronization as follows:

$$\begin{aligned}
&(\text{Master system})\\
&\begin{pmatrix}\dot{x}_1(t)\\ \dot{y}_1(t)\\ \dot{z}_1(t)\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a & -b & -c\end{pmatrix}\begin{pmatrix}x_1(t)\\ y_1(t)\\ z_1(t)\end{pmatrix} - \begin{pmatrix}0\\ 0\\ x_1^3(t)\end{pmatrix}\\
&(\text{Slave system})\\
&\begin{pmatrix}\dot{x}_2(t)\\ \dot{y}_2(t)\\ \dot{z}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a & -b & -c\end{pmatrix}\begin{pmatrix}x_2(t)\\ y_2(t)\\ z_2(t)\end{pmatrix} - \begin{pmatrix}0\\ 0\\ x_2^3(t)\end{pmatrix} + \begin{pmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{pmatrix}
\end{aligned} \tag{2}$$

where $[x_1(t), y_1(t), z_1(t)]^T \in R^3$ and $[x_2(t), y_2(t), z_2(t)]^T \in R^3$ are the states variables; a, b, and c are the parameters of the master and slave systems (2), respectively; and $\boldsymbol{u}(t) = [0, 0, u_3(t)]^T$ is the feedback controller vector.

Robust Synchronization Between Two Identical Coullete Chaotic Systems with Unknown Time-Varying Parameters

In this subsection, the problem statement for the robust synchronization of two identical uncertain Coullete chaotic systems with unknown time-varying parameters, unknown model uncertainties, and unknown external disturbances is given. Thus, the adaptive synchronization for the two nearly identical Coullete chaotic systems coupled in a master-slave system is formulated as follows:

$$\begin{aligned}
&(\text{Master system})\\
&\begin{pmatrix}\dot{x}_1(t)\\ \dot{y}_1(t)\\ \dot{z}_1(t)\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a & -b & -c\end{pmatrix}\begin{pmatrix}x_1(t)\\ y_1(t)\\ z_1(t)\end{pmatrix} - \begin{pmatrix}0\\ 0\\ x_1^3(t)\end{pmatrix} + \begin{pmatrix}f_1(x_1(t))\\ f_2(y_1(t))\\ f_3(z_1(t))\end{pmatrix} + \begin{pmatrix}D_1(t)\\ D_2(t)\\ D_3(t)\end{pmatrix}\\
&(\text{Slave system})\\
&\begin{pmatrix}\dot{x}_2(t)\\ \dot{y}_2(t)\\ \dot{z}_2(t)\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ a_1(t) & -b_1(t) & -c_1(t)\end{pmatrix}\begin{pmatrix}x_2(t)\\ y_2(t)\\ z_2(t)\end{pmatrix} - \begin{pmatrix}0\\ 0\\ x_2^3(t)\end{pmatrix} + \begin{pmatrix}g_1(x_2(t))\\ g_2(y_2(t))\\ g_3(z_2(t))\end{pmatrix}\\
&\qquad + \begin{pmatrix}d_1(t)\\ d_2(t)\\ d_3(t)\end{pmatrix} + \begin{pmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{pmatrix}
\end{aligned} \tag{3}$$

where $[x_1(t), y_1(t), z_1(t)]^T \in R^3$ and $[x_2(t), y_2(t), z_2(t)]^T \in R^3$ are the states variables of the master and slave systems (3), respectively; a, b, and c are the known parameters of the master system in (3); and $a_1(t)$, $b_1(t)$, and $c_1(t)$ are the unknown time-varying parameters of the slave system in (3), which needs to be estimated. $f_i(.)$ and $g_i(.)$ for $i = 1, 2, 3$

are the unknown model uncertainties and $D_i(.)$ and $d_i(.)$ for $i = 1, 2, 3$ are the unknown external disturbances present in the master and slave systems (3), respectively; and $\boldsymbol{u}(t) = [u_1(t), u_2(t), u_3(t)]^T$ is the feedback controller vector.

Solution

Solution to Problem 2.2.1

The matrix of error system for the synchronization between two nearly identical Coullete chaotic systems (2) is given as follows:

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_1^3(t) - x_2^3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_3(t) \end{pmatrix} \tag{4}$$

where $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$, and $e_3(t) = z_2(t) - z_1(t)$ are the synchronization errors.

Theorem 1. *If the control input vector is designed such that*

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_2^3(t) - x_1^3(t) \end{pmatrix} - \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} \tag{5}$$

where the sub-controller matrix $\boldsymbol{v}(t)$ *in Eq.* (5) *:*

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \tag{6}$$

where k_{ij}, $[i, j = 1, 2, 3]$ *is an LCP matrix, and then, the two coupled chaotic systems* (2) *are globally exponentially synchronized.*

Proof of Theorem 1. Substituting Eqs. (5) and (6) into Eq. (4) gives

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix} - \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \right) \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}. \tag{7}$$

Since $u_1(t) = u_2(t) = 0$, therefore, $v_1(t) = v_2(t) = 0$ and Eq. (7) becomes

$$\begin{aligned} \begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} &= \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \right) \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a - k_{31} & -(b + k_{32}) & -(c + k_{33}) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}. \end{aligned} \tag{8}$$

Note that the obtained linearized error system (8) is in the form of $\dot{e}(t) = Ae(t)$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a - k_{31} & -(b + k_{32}) & -(c + k_{33}) \end{pmatrix}. \tag{9}$$

Thus, the remaining problem is that the LCPs k_{31}, k_{32}, and k_{33} are chosen such that the real parts of all eigenvalues of the matrix $A \in R^{3 \times 3}$ in (9) are negative with suitable positions of the LCPs in a complex plane, with fast and smooth convergence of the synchronization error signals. In these circumstances (9), if the LCPs satisfy the following condition:

$$k_{31} > a,\ k_{32} \geq 0 \text{ and } k_{32} \geq k_{33} \geq 0, \tag{10}$$

then, by the Routh-Hurwitz criterion and Lyapunov stability theory, the closed-loop system (8) is globally exponentially stable. Therefore, the two coupled chaotic systems (2) are globally exponentially synchronized.

Remark 2. *In the following subsection, this study finds numerically the correct balance between the convergence rates of the synchronization error signals to the origin and magnitude of the suitable LCPs.*

Numerical Simulation Results and Discussion

The parameters of the Coullete chaotic system (1) are set as $a = 5.5$, $b = 3.5$, and $c = 1$, while the initial values of the state vectors are taken as $x_1(0) = 0.145$, $y_1(0) = 0.625$, $z_1(0) = 0.925$ and $x_2(0) = 0.945$, $y_2(0) = 0.032$, $z_2(0) = 0.112$, respectively. The corresponding numerical results are given as follows.

Case 1: From matrix A in Eq. (9) and the condition (10), it has been observed that the stability of the closed-loop system (8) depends on the magnitude of the LCP k_{31} and the magnitude of k_{31} depends on the magnitude of the LCPs k_{32} and k_{33}. Therefore, let us fix $k_{32} = k_{33} = 3$ and optimize k_{31}.

If $k_{31} = 5.5$, then, the poles of the linearized error system (8) are $\{-1 \pm 1.87083\, i,\ 0\}$. One can notice that one of the eigenvalues is positive. Hence, the closed-loop system (8) is unstable, which is also confirmed from Fig. 2a.

For $k_{31} = 5.6$, 6, 9, 15, 25, 31, and 31.49, the closed-loop system (8) is globally exponentially stable with the following corresponding poles: $\{-1.99 \pm 1.59i,\ -0.015\}$, $\{-1.96 \pm 1.58i,\ -0.08\}$, $\{-1.5 \pm 1.12i,\ -1\}$, $\{-0.56 \pm 1.73i, -2.88\}$, $\{-0.162 \pm 2.298i,\ -3.68\}$, $\{-0.01 \pm 2.53i,\ -3.98\}$ and $\{-0.02 \pm 2.58i,\ -4.04\}$, respectively, which can be confirmed from Fig. 2b–f.

If $k_{31} = 31.5$, then, the poles of the linearized error system (8) are $\{1.387 \times 10^{-15} \pm 2.55i,\ -4\}$. Hence, the closed-loop system (8) is unstable, which is also confirmed from Fig. 2g.

Thus, for $5.5 < k_{31} < 31.5$, the closed-loop system (8) is globally exponentially stable and the perfect synchronization behavior is achieved at $k_{31} = 9$ after $t \approx 3$ s, with underdamped oscillation as shown in Fig. 2c.

Case 2: Let us fix $k_{32} = k_{33} = 1$ and optimize k_{31}. Then, from the numerical study similar as above, it is observed that the closed-loop system (8) is globally exponentially stable at $5.5 < k_{31} < 14.5$ and the perfect synchronization behavior is achieved at $k_{31} = 8$ after $t \approx 6$ s, with underdamped oscillation as shown in Fig. 3.

Comparative Study

The developed active synchronization controller approach has advantages over the past studies in [16] in terms of the control effort, synchronization transient speed, and suitable position of the LCPs in a complex plane for the GES. For example, in terms of the control effort, only one input feedback controller (5) is utilized to accomplish the GES, while in [16], three control functions are designed. Similarly, in this study, the

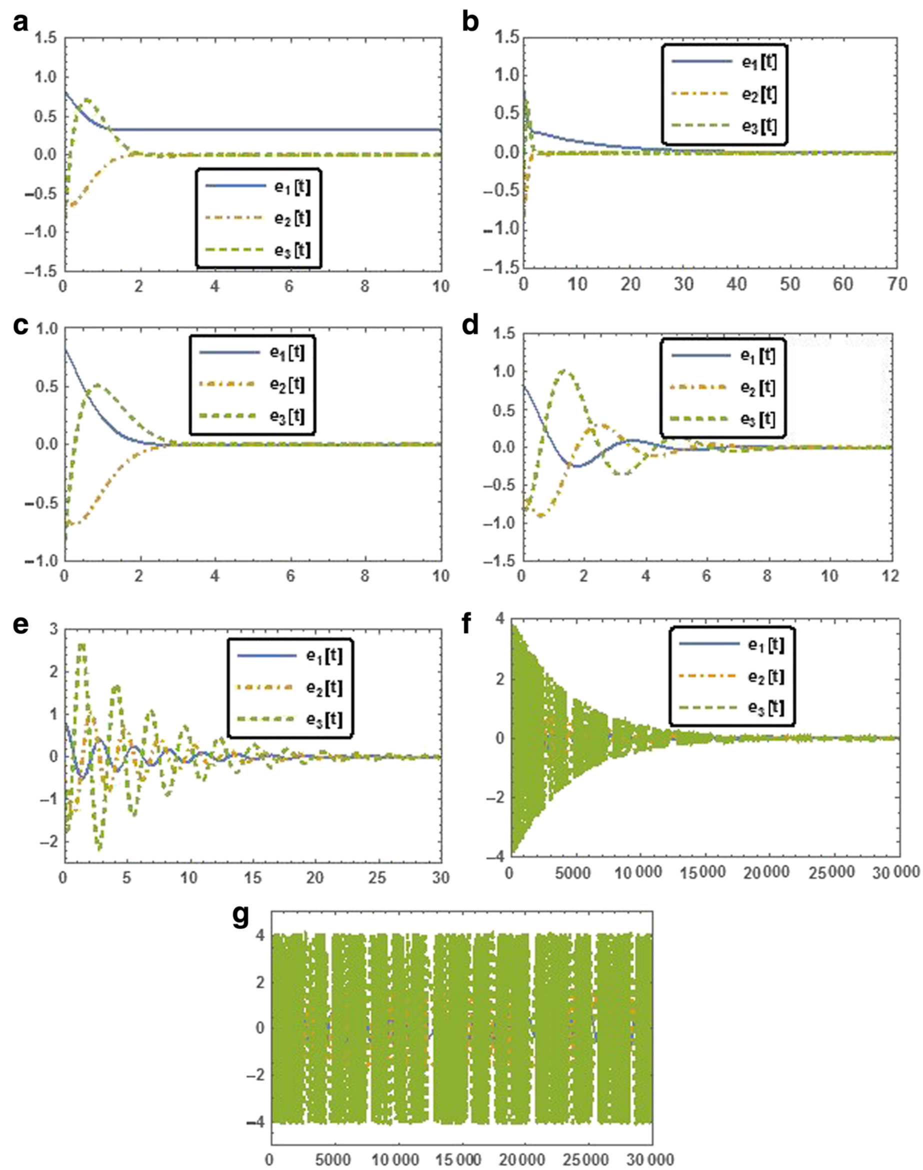

Fig. 2 Time series of the error signals when **a** $k_{31} = 5.5$, **b** $k_{31} = 6$, **c** $k_{31} = 9$, **d** $k_{31} = 15$, **e** $k_{31} = 25$, **f** $k_{31} = 31.49$, and **g** $k_{31} = 31.5$

synchronization speed is 3 s (Fig. 2c), whereas in [15], the synchronization speed is 5 s. Thus, the time difference is 2 s. Furthermore, the proposed AC approach (5) also identifies the correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the LCPs for a fast and smooth synchronization.

The proposed AC function (5) contains a partially nonlinear term and a feedback term. The present study does not only improve the synchronization speed and quality but also decreases the number of feedback controllers. This considerably reduces the amount of energy for the chaos synchronization and establishes the GES. These features give advantages of the current study over the past published works in the literature concerned.

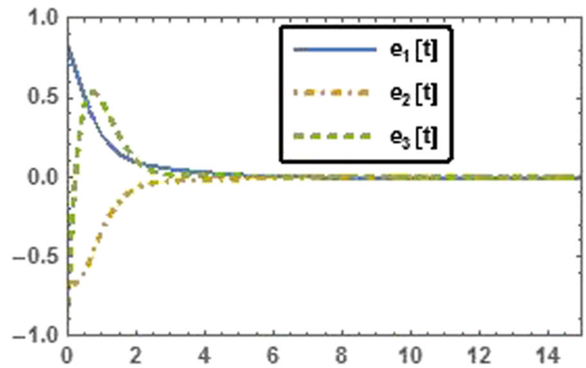

Fig. 3 Time series of the error signals when $k_{31} = 8$

Solution to Problem 2.2.2

The matrix of the error system for the adaptive robust synchronization between two nearly identical Coullete chaotic systems (3) is given as follows:

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -c \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -x_1(t)x_2(t) - \left(x_1^2(t) + x_2^2(t)\right) & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -e_a(t) & e_b(t) & e_c(t) \end{pmatrix} \begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} g_1(x_2(t)) - f_1(x_1(t)) \\ g_2(y_2(t)) - f_2(y_1(t)) \\ g_3(z_2(t)) - f_3(z_1(t)) \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \tag{11}$$

where $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$ and $e_3(t) = z_2(t) - z_1(t)$ are the synchronization errors and $e_a(t) = a - a_1(t), e_b(t) = b - b_1(t)$, and $e_c(t) = c - c_1(t)$ are the estimation of unknown time-varying parameters. Note that $\dot{e}_a(t) = -\dot{a}_1(t),\ \dot{e}_b(t) = -\dot{b}_1(t)$, and $\dot{e}_c(t) = -\dot{c}_1(t)$. The adaptive synchronization of two coupled chaotic systems (3) is accomplished in the sense that:

$$\lim_{t\to\infty} \|e_i(t)\| = 0, \quad i = 1,\ 2,\ 3, \tag{12}$$

and the unknown time-varying parameters are estimated from the system parameters in the sense that:

$$\lim_{t\to\infty} |e_a(t)| = \lim_{t\to\infty} |a - a_1(t)| = 0, \quad \lim_{t\to\infty} |e_b(t)| = \lim_{t\to\infty} |b - b_1(t)| \text{ and } \lim_{t\to\infty} |e_c(t)| = \lim_{t\to\infty} |c - c_1(t)| = 0\,. \tag{13}$$

Assumption 1 [33]. *It is assumed that the unknown model uncertainties and external disturbances are bounded. Therefore, there exist unknown positive constants* Δ_i^m *and* Δ_i^s *such that*

$$|f_i(.)| \le \Delta_i^m \text{ and } |g_i(.)| \le \Delta_i^s, \quad \text{for } i = 1,\ 2,\ 3.$$

$$|g_i(.) - f_i(.)| \le \beta_i, \quad \text{for } i = 1,\ 2,\ 3, \tag{14}$$

where β_1 *is any unknown positive constant such that* $\beta_i = \Delta_i^m + \Delta_i^s$.

Assumption 2 [34]. *Let $B \subset R^n$ be a bounded region containing the whole attractor of the chaotic (or hyperchaotic) system, such that no signal of the chaotic (or hyperchaotic) system ever leaves it. Then, there exist positive constants $B_x \in R$, $B_y \in R$ and $B_z \in R$, such that*

$$|x(t)| \leq B_x,\ |y(t)| \leq B_y,\ and\ |z(t)| \leq B_z. \tag{15}$$

Theorem 2. *If the control input $u_i(t)$, $i = 1, 2, 3$, in* (3) *is designed such that*

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} -\hat{k}_1(\exp(-\eta|e_1(t)|)) & 0 & 0 \\ 0 & -\hat{k}_2(\exp(-\eta|e_2(t)|)) & 0 \\ x_1^2(t) + x_2^2(t) & 0 & -\hat{k}_3(\exp(-\eta|e_3(t)|)) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} - \begin{pmatrix} \beta_1 sgn(e_1(t)) \\ \beta_2 sgn(e_3(t)) \\ \beta_3 sgn(e_3(t)) \end{pmatrix}, \tag{16}$$

and the unknown time-varying parameters $a_1(t)$, $b_1(t)$ and *$c_1(t)$ are estimated by the following adaptation laws:*

$$\dot{a}_1(t) = -x_2(t)e_3(t),\ \dot{b}_1(t) = y_2(t)e_3(t),\ \text{and}\ \dot{c}_1(t) = z_2(t)e_3(t). \tag{17}$$

where η is any positive constant; exp and sgn, respectively, denote the exponential and signum functions; and $\hat{k}_i$, for *$i = 1, 2, 3$ is the estimated LCP, which is updated according to the following adaptation algorithm:*

$$\dot{\hat{k}}_i(t) = -\rho(\exp(-\eta|e_i(t)|))e_i^2(t),\ \ \hat{k}_i(0) = 0,\ \text{for}\ i = 1, 2, 3, \tag{18}$$

where ρ is any positive real constant determining the adaptation process. Then, the two coupled chaotic systems (3) *are asymptotically synchronized.*

Proof of Theorem 2. Substituting Eq. (16) into Eq. (11) gives

$$\begin{aligned} \begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} &= \begin{pmatrix} -\hat{k}_1(\exp(-\eta|e_1(t)|)) & 1 & 0 \\ 0 & -\hat{k}_2(\exp(-\eta|e_2(t)|)) & 1 \\ a & -b & -c-\hat{k}_3(\exp(-\eta|e_3(t)|)) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -x_1(t)x_2(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -e_a(t) & e_b(t) & e_c(t) \end{pmatrix} \begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} \\ &+ \begin{pmatrix} g_1(x_2(t))-f_1(x_1(t)) \\ g_2(y_2(t))-f_2(y_1(t)) \\ g_3(z_2(t))-f_3(z_1(t)) \end{pmatrix} + \begin{pmatrix} d_1(t)-D_1(t) \\ d_2(t)-D_2(t) \\ d_3(t)-D_3(t) \end{pmatrix} - \begin{pmatrix} \beta_1 sgn(e_1(t)) \\ \beta_2 sgn(e_3(t)) \\ \beta_3 sgn(e_3(t)) \end{pmatrix}. \end{aligned} \tag{19}$$

Consider a Lyapunov function as follows:

$$V\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} = \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T P \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \left(\hat{k}_1(t)-k_1\right)^2 \\ \left(\hat{k}_2(t)-k_2\right)^2 \\ \left(\hat{k}_3(t)-k_3\right)^2 \end{pmatrix} + \frac{1}{2}\left(e_a^2(t) + e_b^2(t) + e_c^2(t)\right) \geq 0, \tag{20}$$

where

$$P = diag\left[\frac{1}{2},\ \frac{b}{2},\ \frac{1}{2}\right]. \tag{21}$$

The time derivative of Eq. (20) is given as:

$$\dot{V}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} = \begin{pmatrix} e_1(t) & 0 & 0 \\ 0 & be_2(t) & 0 \\ 0 & 0 & e_3(t) \end{pmatrix}\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{pmatrix} - \begin{pmatrix} e_a(t) & 0 & 0 \\ 0 & e_b(t) & 0 \\ 0 & 0 & e_c(t) \end{pmatrix}\begin{pmatrix} \dot{e}_a(t) \\ \dot{e}_b(t) \\ \dot{e}_c(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \left(\hat{k}_1(t)-k_1\right)\dot{\hat{k}}_1(t) \\ \left(\hat{k}_2(t)-k_2\right)\dot{\hat{k}}_2(t) \\ \left(\hat{k}_3(t)-k_3\right)\dot{\hat{k}}_3(t) \end{pmatrix}. \tag{22}$$

Using Eq. (19) into Eq. (22) yields:

$$\begin{aligned}\dot{V}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} = &\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} -\hat{k}_1(\exp(-\eta|e_1(t)|)) & \frac{1}{2} & \frac{a-|x_1(t)||x_2(t)|}{2} \\ \frac{1}{2} & -\hat{k}_2(\exp(-\eta|e_2(t)|)) & 0 \\ \frac{a-|x_1(t)||x_2(t)|}{2} & 0 & -c-\hat{k}_3(\exp(-\eta|e_3(t)|)) \end{pmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} g_1(x_2(t))-f_1(x_1(t))+d_1(t)-D_1(t) \\ g_2(y_2(t))-f_2(y_1(t))+d_2(t)-D_2(t) \\ g_3(z_2(t))-f_3(z_1(t))+d_3(t)-D_3(t) \end{pmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} - \begin{pmatrix} \beta_1 sgn(e_1(t)) \\ \beta_2 sgn(e_3(t)) \\ \beta_3 sgn(e_3(t)) \end{pmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -e_a(t) & e_b(t) & e_c(t) \end{pmatrix}\begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \left(\hat{k}_1(t)-k_1\right)\dot{\hat{k}}_1(t) \\ \left(\hat{k}_2(t)-k_2\right)\dot{\hat{k}}_2(t) \\ \left(\hat{k}_3(t)-k_3\right)\dot{\hat{k}}_3(t) \end{pmatrix} \\ &- e_a(t)\dot{a}_1(t)-e_b(t)\dot{b}_1(t)-e_c(t)\dot{c}_1(t)\end{aligned}$$

$$\begin{aligned}\leq &\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} -\hat{k}_1(\exp(-\eta|e_1(t)|)) & \frac{1}{2} & \frac{a-x_1(t)x_2(t)}{2} \\ \frac{1}{2} & -\hat{k}_2(\exp(-\eta|e_2(t)|)) & 0 \\ \frac{a-x_1(t)x_2(t)}{2} & 0 & -c-\hat{k}_3(\exp(-\eta|e_3(t)|)) \end{pmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} \left|g_1(x_2(t))-f_1(x_1(t))+d_1(t)-D_1(t)\right| sgn(e_1(t))-\beta_1 sgn(e_1(t)) \\ \left|g_2(y_2(t))-f_2(y_1(t))+d_2(t)-D_2(t)\right| sgn(e_3(t))-\beta_2 sgn(e_3(t)) \\ \left|g_3(z_2(t))-f_3(z_1(t))+d_3(t)-D_3(t)\right| sgn(e_3(t))-\beta_3 sgn(e_3(t)) \end{pmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \left(\hat{k}_1(t)-k_1\right)\dot{\hat{k}}_1(t) \\ \left(\hat{k}_2(t)-k_2\right)\dot{\hat{k}}_2(t) \\ \left(\hat{k}_3(t)-k_3\right)\dot{\hat{k}}_3(t) \end{pmatrix} - e_a(t)x_2(t)e_3(t) + e_b(t)y_2(t)e_3(t) \\ &+ e_c(t)z_2(t)e_3(t) - e_a(t)\dot{a}_1(t)-e_b(t)\dot{b}_1(t)-e_c(t)\dot{c}_1(t).\end{aligned} \tag{23}$$

Using Assumption 1 and the fact that $|sgn(e_i)| \le 1$ for $i = 1, 2, 3$, in Eq. (23) yields:

$$\le \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} -\rho\hat{k}_1(t)(\exp(-\eta|e_1(t)|)) & \frac{1}{2} & \frac{a-B_x}{2} \\ \frac{1}{2} & -\rho\hat{k}_2(t)(\exp(-\eta|e_2(t)|)) & 0 \\ \frac{a-B_x}{2} & 0 & -c+\rho\hat{k}_3(\exp(-\eta|e_3(t)|)) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \left(\hat{k}_1(t)-k_1\right)\dot{\hat{k}}_1(t) \\ \left(\hat{k}_2(t)-k_2\right)\dot{\hat{k}}_2(t) \\ \left(\hat{k}_3(t)-k_3\right)\dot{\hat{k}}_3(t) \end{pmatrix} -e_a(t)(\dot{a}_1(t) + x_2(t)e_3(t))-e_b(t)\left(\dot{b}_1(t)-y_2(t)e_3(t)\right)$$
$$- e_c(t)(\dot{c}_1(t)-z_2(t)e_3(t)). \tag{24}$$

Using the parameter update laws (17) and (18) into Eq. (24) gives:

$$\dot{V}\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \le - \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} \rho k_1(t)(\exp(-\eta|e_1(t)|)) & -\frac{1}{2} & \frac{B_x-a}{2} \\ -\frac{1}{2} & \rho k_2(t)(\exp(-\eta|e_2(t)|)) & 0 \\ \frac{B_x-a}{2} & 0 & c+\rho k_3(\exp(-\eta|e_3(t)|)) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}$$

$$\dot{V}(\boldsymbol{e}(t)) \le -\boldsymbol{e}(t)^T Q\, \boldsymbol{e}(t) \le 0, \tag{25}$$

where $\boldsymbol{e}(t) = [|e_1(t)|,\ |e_2(t)|,\ |e_3(t)|]^T$ is the absolute state error vector, and

$$Q = \begin{pmatrix} \rho k_1(t)(\exp(-\eta|e_1(t)|)) & -\frac{1}{2} & \frac{B_x-a}{2} \\ -\frac{1}{2} & \rho k_2(t)(\exp(-\eta|e_2(t)|)) & 0 \\ \frac{B_x-a}{2} & 0 & c+\rho k_3(\exp(-\eta|e_3(t)|)) \end{pmatrix}. \tag{26}$$

At this stage, the remaining problem is that if the estimate of the LCPs k_1, k_2, and k_3 and the two positive constants η and ρ are chosen such that the matrix, $Q \in R^{3\times 3}$ (26) becomes a positive definite matrix (PDM). Since $V(\boldsymbol{e}(t))$ is positive definite, then the equilibrium point $\left(e_i(t) = 0,\ \hat{k}_i = k_i,\ \ i = 1,\ 2,\ 3\right)$ of the systems (11) and (18) is asymptotically stable. Therefore, the two coupled chaotic systems (3) are asymptotically synchronized. This completes the proof of Theorem 2.

Numerical Simulation Results and Discussion

Numerical simulation results are furnished in order to verify the robustness and performance of the proposed ASC approach. The true value of the parameters of the Coullete chaotic system (1) are set as $a = 5.5$, $b = 3.5$, and $c = 1$, and these values are unknown to the slave system in (3). The initial values of the states vectors are taken as $x_1(0) = 0.145$, $y_1(0) = 0.625$, $z_1(0) = 0.925$ and $x_2(0) = 0.945$, $y_2(0) = 0.032$, $z_2(0) = 0.112$, respectively. The estimated absolute values of the state vectors of the Coullete chaotic system (1) are $B_x \le 3.6$, $B_y \le 6$, and $B_z \le 12$ through numerical simulation. The controlling parameters are considered as $k_1 = k_2 = 5$ and $k_3 = 10$, and the two positive constants η and ρ are taken as $\eta = 0.01$ and $\rho = 1$. In numerical simulations, the following model uncertainties and external disturbances are applied to the master and slave systems (3), respectively.

$$\begin{aligned}
&\text{(Master system)}\\
&f_1(x_1(t)) + D_1(t) = 0.3\sin\left(\frac{\pi}{3}x_1(t)\right) - 0.01\cos(10t),\\
&f_2(y_1(t)) + D_2(t) = -0.25\cos\left(\frac{\pi}{4}y_1(t)\right) - 0.03\sin(20t),\\
&f_3(z_1(t)) + D_3(t) = 0.3\sin\left(\frac{\pi}{2}z_1(t)\right) + 0.04\cos(10t),\\
&\text{(Slave system)}\\
&g_1(x_2(t)) + d_1(t) = -0.4\cos\left(\frac{\pi}{6}x_2(t)\right) + 0.02\sin(30t),\\
&g_2(y_2(t)) + d_2(t) = 0.25\sin\left(\frac{5\pi}{6}y_2(t)\right) - 0.01\cos(20t),\\
&g_3(z_2(t)) + d_3(t) = 0.15\cos\left(\frac{2\pi}{3}z_2(t)\right) - 0.01\cos(15t).
\end{aligned} \tag{27}$$

Accordingly, $\beta_1 = 0.64$, $\beta_2 = 0.54$, and $\beta_3 = 0.5$.

The corresponding numerical simulation results are given as follows:

Figure 4 displays the result of the synchronized error signals. It is demonstrated that the error signals (11) completely synchronize within a short transient time $t \approx 0.7\ s$ in the presence of external disturbances and model uncertainties under the control action (16). The adaptive process of parameters is shown in Fig. 5. From Fig. 5, it can be observed that the unknown time-varying parameters $a_1(t) = a + 0.2\sin(35t)$, $b_1(t) = b + 0.1\sin(25t)$, and $c_1(t) = c + 0.02\cos(90t)$ with initial values $a_1(0) = 7$, $b_1(0) = 2$, $c_1(0) = -1$, converge to the true values of a, b, and c as $t \to \infty$, under the parameter update laws (20). Figure 6 shows the time history of the input control signals. The proposed ASC approach (16) is robust against different types of perturbations. The controller response time is short, and the error signals converge to the origin with critically damped oscillation.

Remark 3.

(i) *The proposed ASC approach* (16) *contains the linear terms, some partially nonlinear terms, and a feedback term. The exponential term* $(exp(-\eta|e_i(t)|))$ *for* $i = 1, 2, 3$, *in the controller* (16) *provides smoothness to the error signals with small amplitude and without disturbing the convergence property, even in the presence of unknown external disturbance and model uncertainties.*

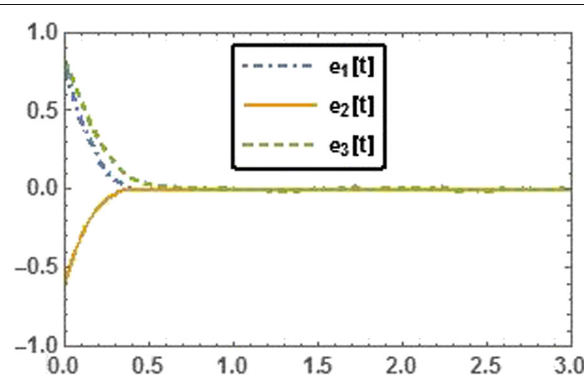

Fig. 4 Time series of the synchronized error states

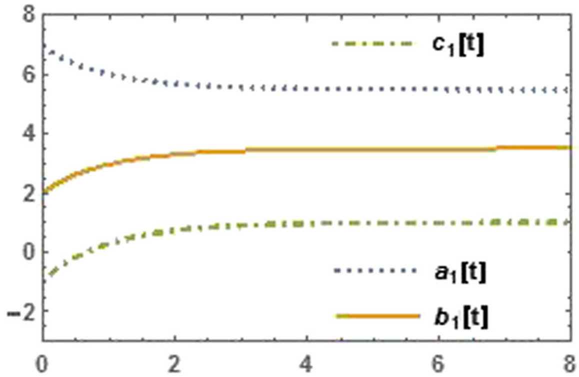

Fig. 5 Time series of the estimated parameter $a_1(t)$, $b_1(t)$, and $c_1(t)$

(ii)*By selecting a smaller value of η providing the fast convergence rates of the error signals to the origin.*

Remark 4. *The proposed ASC approach can be easily utilized for the complete and generalized synchronization for a class of chaotic as well as hyperchaotic systems. For example, the synchronization of two identical hyperchaotic Lu systems* [20, 22] *can also be achieved by applying the following nonlinear control function:*

$$u_i(t) = -k_i(exp(-\eta|e_i(t)|)),\ for\ \ i = 2,\ 4,\ and\ u_i(t) = 0,\ for\ \ i = 1,\ 3, \tag{28}$$

and the parameter update laws are given as:

$$\begin{aligned}\dot{a}_1(t) &= -(y_2(t) - x_2(t))e_1(t),\ \dot{b}_1(t) = z_2(t)e_3(t),\ \dot{c}_1(t)\\ &= -y_2(t)e_2(t),\ and\ \ \dot{d}_1(t) = w_2(t)e_4(t),\end{aligned} \tag{29}$$

where the true values of parameters are taken as $a = 15$, $b = 5$, $c = 10$, *and* $d = 1$.

Numerical simulation results are shown in Figs. 7 *and* 8. *As compared to the past published works* [20, 22], *the synchronization speed is faster* (0.6 *s vs* 4 *s*). *Moreover, the synchronized error signals in* [20, 22] *converged to the origin with underdamped oscillation, while in the current study, the synchronized error signals converged to the*

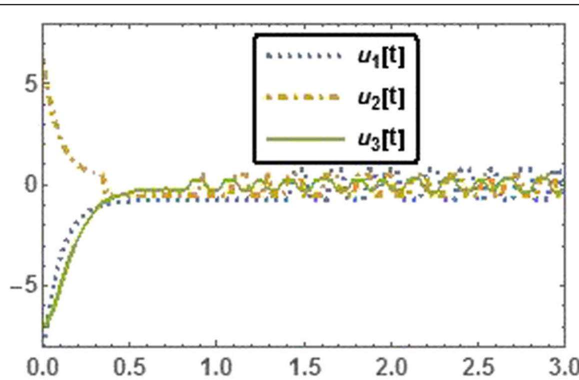

Fig. 6 Time series of the control signals

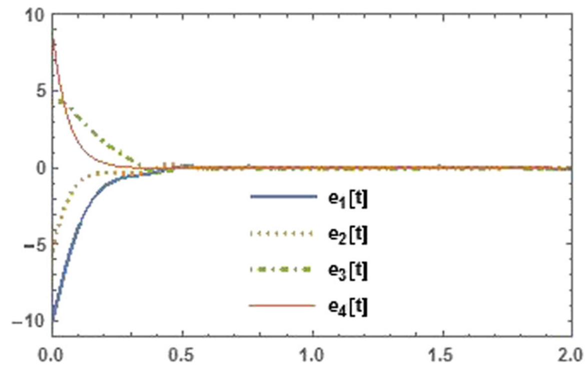

Fig. 7 Time series of the synchronized error states for two

origin with critically damped oscillation, which shows the less amount of energy utilized for CS objective.

As a matter of fact, the time variance property of the system parameters and the existence of total disturbances in problem formulation and the controller design procedure for the robust stability of the closed-loop system are making the proposed ASC approach to be more effective as compared to the previous results and can be easily implemented in practice.

Conclusions

In this paper, two synchronization control strategies were proposed for the CS objective. The active controller design procedure was based on the Routh-Hurwitz criterion. A single input active controller was proposed which established the globally exponential CS with comparatively low energy. Similarly, the correct balance between the synchronized error convergence rates to the origin and magnitude of the linear controlling parameters was identified. Accordingly, a novel adaptive-based synchronization controller was proposed and suitable adaptive laws of time-varying parameters were designed which accomplished the robust CS in a short time. The closed-loop stability for CS was proved and the effectiveness of the proposed algorithm schemes was assessed by numerical simulations and by comparing with past published works.

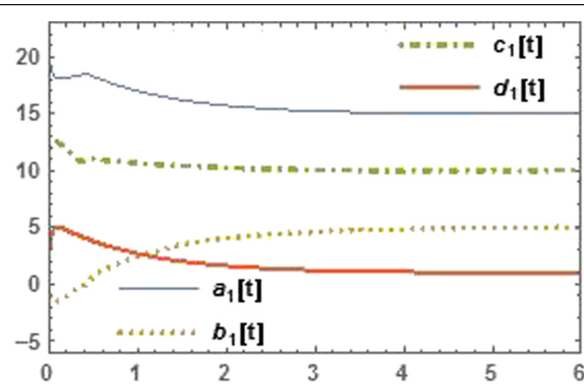

Fig. 8 Time series of the estimated parameters $a_1(t)$, identical uncertain hyperchaotic systems ($k_2 = k_4 = 15$) $b_1(t)$, $c_1(t)$, and $d_1(t)$

Limitations of the proposed AC and ASC algorithm approaches are summarized as follows.

(i) In the "Solution" section, in using the active control strategy and linearizing the error system, it has been shown that the correct balance between the converging rates of the synchronization error signals to the origin and magnitude of the suitable linear controlling parameters is identified computationally for two identical Coullete chaotic systems using a single control function. The structure of the Coullete chaotic system is different from that of the usual chaotic or hyperchaotic systems such as the Lorenz chaotic system, the Chen chaotic system, and the Rossler hyperchaotic system. Therefore, a generalized analytic approach should be investigated to discuss the same problem for the complete and generalized synchronization of a general class of chaotic (or hyperchaotic) systems.

(ii) As shown in the "Solution to Problem 2.2.2" section, by using the ASC approach, the synchronized error signals converged to the origin asymptotically in a short time in spite of different types of uncertainties. This approach can be applied to various chaotic as well hyperchaotic systems. The main issue is concerned with the amount of control signals. In the case of generalized synchronization, the control inputs and bounds may become too large.

(iii) The proposed ASC approaches can be applied only to continuous time dynamical chaotic systems.

Additional file

Additional file 1: Mathematical codes. (DOC 51 kb)

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Authors' Contributions
This work was carried out in collaboration among all authors. IA proposed the main idea, performed the literature review, and suggested the problem. ABI designed the study and performed the mathematical analysis. ABS performed the numerical simulations. IA and ABI revised the manuscript critically for important intellectual content. All the authors read and approved the final manuscript to be published in JUAA.

Competing Interests
The authors declare that they have no competing interests.

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