Bhattacharya and Roychowdhury *Journal of Uncertainty Analysis and Applications* 2013, **1**:14
http://www.juaa-journal.com/content/1/1/14

Journal of Uncertainty Analysis and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Optimal redundancy allocation rule for improving system life

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Abstract

Enhancing system life span is an essential area of concern to reliability practitioners. Various techniques can be adopted to achieve this, of which redundancy allocation is a popular one. System life increases whenever a redundant component is added to any of the components of the system, but the amount of increase depends on the choice of component to which redundancy is allocated. We need to identify the best choice of component, to which when a redundant component is added, maximum increase in system life is achieved. In this paper, we propose a general rule for maximizing system life stochastically by properly allocating a redundant component. Knowledge about the order of component lives is enough to make the optimal decision here. This rule can be applied to any simple or complex systems, and any number of redundant components can be added to the system to have maximum possible improvement. Some useful coherent systems are studied here in the light of derived results.
MSC 2010: Primary 90B25; 62 N05; 60E15; Secondary 60 K10; 65 K10; 62G30

Keywords: Coherent system; Redundancy allocation; Stochastic ordering; System life

Introduction

Designing a system to perform its intended job at least up to a specified time, known as a target life or mission time, always has been a challenge to reliability practitioners. To enhance the random system life by allocating redundant components is a common practice. However, that does not ensure stochastic maximization of system life unless the redundancy is allocated properly. There has been no general rule for addressing this demanding issue of optimal allocation of redundancy that can be applied to any simple or complex coherent systems which maximizes the life span of a system stochastically. There may be a situation where no compromise with meeting the target life can be afforded. In such cases, the system needs to be so designed that it can operate satisfactorily at least for some specified length of time. Mention may be made of a missile flight, which would be required to function up to the time it takes in reaching its known destination or artificial satellites, and space explorers have to be safe for a predetermined mission time without any major failure. Another example could be military communication equipment, which must last all through its field operation of known length. Thus, designing a reliable system often requires that the system performs its intended job of specific duration without any major disruption, and malfunctioning of any of its constituent components may cause a great loss, which may not be restricted only to a loss of money or time due to malfunction.

It is important to note that the amount of increase in system life is different for a different choice of the components to which the redundant component is to be added. Thus, the issue boils down to attaching the redundant components in an effective manner so as to maximize the system life stochastically.

Generally, there are two commonly used types of redundancies - standby redundancy and active or parallel redundancy. In standby redundancy, a redundant component is attached in such a way that it starts functioning immediately after the failure of the component to which it is attached. Parallel redundancy is used when it is difficult or not possible to replace the failed components during operation of the system. The redundant components are connected in parallel with the components of the system, and they function simultaneously with the original components. In this paper, the active or parallel redundancy at component level is considered to increase the system life stochastically. It has been noted in Barlow and Proschan [1] that in the case of active redundancy, component-wise redundancy works much better than the system-wise redundancy.

The past work on reliability optimization can be classified as follows: papers solving the issue by focusing on the enhancement of component life [2-4] and by making provision of adding redundant components [5-8]. In some recent works, Sheikhalishahi et al. [9] presented a reliability redundancy allocation problem where they considered some particular system designs. Cao et al. [10] solved a redundancy allocation problem using a decomposition-based approach in series–parallel systems only. In this paper, we propose a general rule for making an optimal decision to choose a component of a given system to which the redundant component is to be added to stochastically maximize the system life. The method is applicable to any coherent systems in general.

The paper is organized as follows: the present section introduces the work and discusses earlier works in this area. The second section includes preliminaries necessary to develop a method for redundancy allocation, some related definitions, and the result. The third section formulates an optimum allocation rule, using the one which can identify the best choice of component of a system to which the redundant component is to be added to give maximum rise in system life. Applications of the rule to some important coherent systems are included in the fourth section. Finally, the fifth section concludes the work.

Preliminaries and notation

An n-component system is said to be coherent if every component is relevant, i.e., every component has some contribution towards the system performance and if the system is monotone, i.e., the performance of the system improves with the improvement of any component or a subset of components. For the formal definition of an n-component coherent system, one can refer to Barlow and Proschan [1].

A system life can be determined from its component lives. Here we decompose a coherent system in a number of subsystems in such a way that the system fails whenever any of the subsystems fail, and a subsystem fails when all of its components fail. Thus, the system life can be obtained from the subsystem lives.

Next, the concept of stochastic order and stochastic equality of two variables are given below which will be used in the sequel.

Let F and G be two probability distribution functions on real line R. Let two random variables X and Y be distributed according to F and G, respectively. F is stochastically larger than G if

$$F(x) \leq G(x) \text{ for all } x \in R.$$

In this case, we have $P(X \geq t) \geq P(Y \geq t)$ for all t. This means that X exceeds a fixed number with a higher probability than Y does. In other words, X is said to be stochastically larger than Y. In the notation, $X_i \geq_{st} X_j$.

Two random variables X and Y are said to be stochastically equal or simply equal in distribution if

$$P(X \leq t) = P(Y \leq t) \text{ for all } t.$$

It is denoted by $X =_{st} Y$.

In this paper, we develop a rule for selecting a component (optimal solution) for which system life is stochastically maximized, i.e., the chance of increase in system life being maximum is greatest when a redundant component is attached to it. The notion of stochastic ordering of component lives is used to formulate the rule for getting the solution. Let X_1, X_2,..., X_n be independently distributed random lives of the components of an n-component coherent system. By the definition of stochastic ordering of random variables, it can easily be seen that the order of the component lives implies and is implied by the order of component reliabilities, i.e., if reliability of a component is more than that of the other, then it is more likely to have longer life, or vice versa. If X_i be the life of the ith component having reliability $p_i(t)$ at time t, and X_j be the life of the jth component having reliability $p_j(t)$, then

$$\begin{array}{ll} & X_i \geq_{st} X_j \\ \text{if and only if} & P(X_i > t) \geq P(X_j > t), \text{for all } t \\ \text{i.e.,} & p_i(t) \geq p_j(t), \quad \text{for all } t. \end{array} \tag{1}$$

Thus, in case the order of component reliabilities is known, the problem can be solved in a similar manner as will be discussed in this paper.

An implication of stochastic ordering between two random variables is now stated below, which will be used in interpreting the result discussed in Theorem 1 in the next section.

If $X_i \geq_{st} X_j$, then $P(X_i > X_j) \geq P(X_j > X_i)$, i.e., if X_i is stochastically larger than X_j, the chance of X_i being larger than X_j is more, since

$$\begin{aligned} P(X_i > X_j) = \int_{x_j=-\infty}^{\infty} P(X_i > x_j)\, dF_{X_j}(x_j) \geq \int_{x_j=-\infty}^{\infty} P(X_j > x_j)\, dF_{X_j}(x_j), \quad \text{by (1)} \\ = \int_{x_j=-\infty}^{\infty} \left(1 - F_{X_j}(x_j)\right) dF_{X_j}(x_j) = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned} \tag{2}$$

In a group of competing system designs, if the system life for a certain design is stochastically larger than that of the other, the corresponding system reliability will also be more, by similar reasoning as in (1). If T_1 and T_2 are system reliabilities for two different designs, then

$$\begin{aligned} & T_1 \geq_{st} T_2 \\ \text{if and only if} \quad & P(T_1 > t) \geq P(T_2 > t), \text{for all } t. \end{aligned} \tag{3}$$

In Theorem 1, we find the optimal solution that gives stochastically largest system life. By (3), the same solution will also maximize the system reliability.

Throughout the paper, by equality ('=') of random variables (random lives), we mean stochastic equality; by '≥' and '≤' between two random variables, we indicate 'stochastically larger' and 'stochastically smaller', respectively. Similarly, 'smaller (larger)' indicates 'stochastically smaller (larger)', and 'equal' indicates 'stochastically equal'.

Now consider $X_1, X_2,\ldots, X_n$, independently distributed random lives of the components of an n-component system which is decomposed into k subsystems, $C_1, C_2,\ldots, C_k$, of sizes $n_1, n_2,\ldots, n_k$, respectively. Note that $\sum_{i=1}^{k} n_i \geq n$, when the subsystems are overlapping, i.e., one component may belong to more than one subsystem. Non-overlapping subsystems do not share any components. For a system with non-overlapping subsystems, $\sum_{i=1}^{k} n_i = n$.

Let $C_i = \{i_1, i_2, \ldots, i_{n_i}\}$, where i_j is the jth component of the ith subsystem C_i, $j = 1, 2,\ldots, n_i$, $i = 1, 2,\ldots, k$. Let Y_i be the life of the ith subsystem, C_i, i.e., the largest of the lives of all components belonging to C_i.

It is known that a coherent system is always monotone, which indicates that the reliability of a coherent system increases with the improvement of any component or a subset of components. When a redundant component is added to a system component in parallel, the component reliability increases. Thus, the system reliability increases (non-decreases) when a redundant component is added to any of the components of a coherent system. Using redundancy improves system life as well. Let us now prove the following result:

Result

The system life increases (non-decreases) when a redundant component is added to any of the components of the system.

Proof The system life can be written as

$$T = \min_{1 \leq i \leq k} \max_{j \in C_i} X_j = \min_{1 \leq i \leq k} Y_i \tag{4}$$

where $Y_i = \max_{j \in C_i} X_j$ is the life of the ith subsystem, which is the maximum of the component lives of the ith subsystem. The component having a maximum life among the lives of all components in the ith subsystem, by failing, will cause the subsystem to fail. The system life, as given in (4), will be the minimum of the lives of all subsystems.

Let U be the random life of the redundant component, which is independent of $X_1, X_2,\ldots, X_n$.

Suppose $Y_{(1)}$ to be the smallest of the subsystem lives, i.e., $Y_{(1)} = \min_{1 \le i \le k} Y_i$, and $Y_{(2)}$ to be the second smallest. When a redundant component having a random life U is added to a system component, one of the following cases may occur:

Case 1. The redundant component is added to the subsystem with the smallest life, where U will be one of the following:

(i) $U \le Y_{(1)}$
(ii) $Y_{(1)} < U < Y_{(2)}$
(iii) $Y_{(1)} < Y_{(2)} < U$

Case 2. The redundant component is added to some other subsystem (not to the one with the smallest life).

In case 1(i), again $Y_{(1)} = \min_{1 \le i \le k} Y_i$. The system life will remain unchanged.

In 1(ii), $U = \min_{1 \le i \le k} Y_i$. The system life will increase from $Y_{(1)}$ to U.

In 1(iii), $Y_{(2)} = \min_{1 \le i \le k} Y_i$, and the system life will increase from $Y_{(1)}$ to $Y_{(2)}$.

In case 2, there will be no change in system life.

Hence, the result follows.

Our objective is to select the system component which gives maximum rise in the system life when a redundant component is added to it. The next section proves a theorem to get the optimal solution.

Optimum allocation rule

Suppose T_j to be the life of the jth subsystem C_j, if the redundant component is attached to a component belonging to C_j, $j = 1, 2, \ldots, k$.

Now we see how the system life changes if a redundant component (s) is added to a component belonging to C_j.

If s is added to a component belonging to C_j, the life of the jth subsystem will become

$$T_j = Y_j + \max\left(U - Y_j, 0\right), j = 1, 2, \ldots, k. \tag{5}$$

Now suppose $Y_{(1)}, Y_{(2)}, \ldots, Y_{(k)}$ to be the ordered subsystem lives. Then, (4) becomes $T = Y_{(1)}$.

The following result helps select the subsystem to a component of which the redundant component is to be added in order to have maximum system life.

Theorem 1

Increase in system life is stochastically maximum when a redundant component is added to a component belonging to the subsystem with the smallest life.

Proof Here one of the following three cases may occur:

(i) $U \le Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(k)}$
(ii) $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(k)} \le U$
(iii) $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(i)} \le U \le Y_{(i+1)} \le \ldots \le Y_{(k)}, i = 1, 2, \ldots, (k-1)$

Now let us see how system life changes in the three scenarios.

(i) If $U \le Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(k)}$, then by (5), $T_j = Y_j$, for all $j = 1, 2,\ldots, k$, and system life, $T = Y_{(1)}$.
Thus, in this situation, the system life will remain the same no matter which subsystem component the redundant component is added to.

(ii) If $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(k)} \le U$, by (5), $T_j = U$ for all $j = 1, 2,\ldots, k$, and therefore, if s is added to a component belonging to the subsystem having the smallest system life $Y_{(1)}$, the life of that subsystem will then be U, and hence, the system life will be the smallest of other $Y_{(i)}$'s, i.e., $T = Y_{(2)}$, which is greater than $Y_{(1)}$. However, if s is added to a component belonging to any other subsystem, the minimum of subsystem lives will then be $Y_{(1)}$, and hence, the system life will then be $T = Y_{(1)}$.
Thus, in this situation, the optimal choice of subsystem should be the subsystem having the smallest life. The redundant component should be added to a component belonging to the subsystem having the smallest life in order to increase the system life.

(iii) If $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(i)} \le U \le Y_{(i+1)} \le \ldots \le Y_{(k)}$, by (5), $T_j = U$, for all j such that $Y_{(j)} \le U$, and $T_j = Y_{(j)}$, for all j such that $Y_{(j)} \ge U$. Therefore, if s is added to a component belonging to any subsystem having life $\ge U$, the system life will be $T = Y_{(1)}$. If s is added to a component belonging to a subsystem having life $\le U$, but $\ge Y_{(1)}$, then also the system life will be $T = Y_{(1)}$. However, if s is added to a component belonging to a subsystem having the smallest life, the system life will increase to $Y_{(2)}$.
Thus, in this case, also the optimum choice is the subsystem having the smallest life.

Combining all three cases, it is clear that the optimum choice of subsystem is the one that has the smallest life.

The statement 'increase in system life is stochastically maximum' in the theorem is equivalent to the 'chance of increase in system life being maximum is greatest', by the same reasoning as in (2).

Next, we determine which component of the selected subsystem should be chosen in order to get maximum system life. Suppose C_i to be the selected subsystem. Its life is

$$Y_i = \max_{j \in C_i} X_j = X_{(n_i)} \tag{6}$$

where $X_{(1)}, X_{(2)},\ldots, X_{(n_i)}$ are the ordered component lives of subsystem C_i.

Adding s to a subsystem is the same as adding s to any of the n_i components, but since those components may also belong to some other subsystems, we need to take care of this fact while choosing the component to which s will be attached.

If $U \le X_{(n_i)}$, by (6), the subsystem life $Y_i = X_{(n_i)}$.

If $X_{(n_i)} \le U$, by (6), the subsystem life $Y_i = U$.

For a system with overlapping subsystems, the decision of choosing a component is based on other subsystems where the components belong to. We consider the lives of those subsystems that contain the components of the selected subsystem, and find the smallest of them. Following the same logic as in Theorem 1, the corresponding component is the component that maximizes the system life.

Suppose C_i to be the subsystem with the smallest life. Let $C_i = \{i_1, i_2, \ldots, i_{n_i}\}$, where i_1, $i_2, \ldots, i_{n_i}$ are n_i components belonging to C_i. Let the component i_j ($j = 1, 2,\ldots, n_i$) belong to other m_j ($\geq$ 0) subsystems $C_1^{(i_j)}, C_2^{(i_j)}, \ldots, C_{m_j}^{(i_j)}$ (these are nothing but m_j subsystems from C_1, C_2,…, C_k, except C_i), whose lives are, respectively, denoted by $Y_1^{(i_j)}, Y_2^{(i_j)}, \ldots, Y_{m_j}^{(i_j)}$.

Let $Y_{h_j}^{(i_j)}$ be the subsystem life, the smallest among the lives of all subsystems (except C_i) that contain the component i_j, i.e.,

$$Y_{h_j}^{(i_j)} = \min_{1 \leq l \leq m_j} Y_l^{(i_j)}, \quad j = 1, 2, \ldots, n_i$$

For each of components $i_1, i_2, \ldots, i_{n_i}$, we may get one such subsystem. Let their lives be $Y_{h_1}^{(i_1)}, Y_{h_2}^{(i_2)}, \ldots, Y_{h_{n_i}}^{(i_{n_i})}$, respectively. Following the logic used in proving Theorem 1, the component corresponding to the smallest life among $Y_{h_1}^{(i_j)}, Y_{h_2}^{(i_j)}, \ldots, Y_{h_{n_i}}^{(i_j)}$ will be chosen. Thus, we can write the following: Choose component i_j (to which the redundant component s will be attached to achieve maximum increase in system life) if

$$Y_{h_j}^{(i_j)} = \min_{1 \leq q \leq n_j} Y_{h_q}^{(i_q)} \tag{7}$$

Note that in case the subsystem with the second smallest life does not contain any of the components of the subsystem with the smallest life, it does not matter which component of the latter subsystem (one having the smallest life) the redundant component is added to, because in this case, the life of the subsystem with the second smallest life will not change, but the life of the other subsystems may increase when s is added. The system life will then be the minimum of $Y_{(2)}$, the second smallest subsystem life, and max(U, $Y_{(1)}$), the larger between the redundant component life and the smallest subsystem life, i.e., system life $T = \min(Y_{(2)}, \max(U, Y_{(1)}))$.

In case of systems with non-overlapping subsystems, the redundant component can be added to any of the components belonging to the smallest life subsystem. The rise in system life will be the same, and that will be the maximum.

Note that the method discussed above can be used for adding any number of redundant components. The components should be added one at a time.

Application of the rule

Let us first consider some commonly used systems and see how the rule works to identify the component which gives maximum rise in system life when the redundant component is added to it.

In an n-component series system, there are n subsystems, each having a single component. Hence, by Theorem 1, the component (which is also a subsystem, in this case) with the smallest life is to be chosen for adding the redundant component in order to maximize the system life.

For an n-component parallel system, since there is only a single subsystem of size n and none of the components can belong to any other subsystem, the redundant component can be added to any of the components in parallel. The increase in system life will be the same, and that will be the maximum.

Consider a series–parallel system, where components 2 and 3 are in parallel and component 1 is in series to the parallel structure connecting components 2 and 3.

Here, there are two subsystems, {1} and {2, 3}, which are non-overlapping. If life of component 1 is less than that of the larger of components 2 and 3, then by Theorem 1, the redundant component should be added to component 1. Otherwise it should be added to any one of components 2 and 3 to get the maximum system life. Exactly similarly we can solve the problem for a hi-fi system, where components 1 and 2 are in parallel, components 4 and 5 are in parallel, and both of them (parallel structures) are in series with component 3, thus having a series–parallel structure with non-overlapping subsystems.

For a parallel–series system, where components 1 and 2 are in series and component 3 is in parallel to them (that series), the subsystems are {1, 3} and {2, 3}. Without loss of generality, suppose the life of component 1 to be smaller than that of component 2. Then, the life of subsystem {1, 3} will be less. This subsystem has two components, namely, 1 and 3, of which component 3 belongs to another subsystem. Hence, by (7), the redundant component should always be added to component 3, which gives maximum rise in system life.

An application to a bridge system

Let us consider a complex system, like a bridge system, as shown in Figure 1.

Let the order of the component lives be $X_2 < X_4 < X_1 < X_3 < X_5$. The subsystems of this system are {1, 2}, {1, 3, 5}, {2, 3, 4}, and {4, 5}, whose lives are, respectively, X_1, X_5, X_3, and X_5, of which X_1 is the smallest. Hence, by Theorem 1, subsystem {1, 2} is to be chosen. Its components, 1 and 2, also belong to subsystems {1, 3, 5} and {2, 3, 4}, respectively. Since $X_5 > X_3$, the life of subsystem {2, 3, 4} is smaller, and hence, by (7), component 2 is to be chosen to which the redundant component should be added so that the chance of getting maximum rise in system life is greatest.

For example, in particular, if X_1 =5, X_2 =2, $X_3 = 7$, $X_4 = 4$, and $X_5 = 9$, the system life, by (4), is 5. If we add the redundant component having life $U = 8$ to component 2, the system life becomes 8. If we added it to component 1, the system life would then be 7. If we added it to component 3 or 4 or 5, the system life would not have changed then, and that would be 5.

Now suppose the order of the component lives to be $X_2 < X_4 < X_1 < X_5 < X_3$. Then, the subsystem lives are, X_1, X_3, X_3, and X_5, respectively, for subsystems {1, 2}, {1, 3, 5},

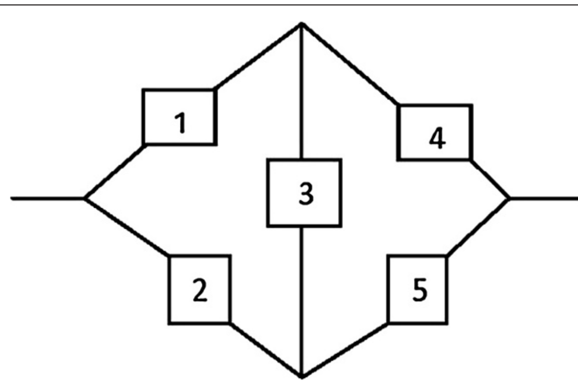

Figure 1 A bridge system.

{2, 3, 4}, and {4, 5}. Hence, the system life is X_1, being the smallest. By the theorem, we choose subsystem {1, 2}. Now since the lives of {1, 3, 5} and {2, 3, 4} are equal, the redundant component can be added to any one of components 1 and 2, and in each case, it will be the same and the increase in system life (and hence the system life) is stochastically maximum.

Conclusions and discussion

In this paper, we have developed a rule for selecting a component of a coherent system to which we can attach a redundant component to maximize the system life stochastically. If we have an idea about the order of the component lives, the rule proposed here can be used to make an optimal decision to choose the component. This rule can be applied to any simple or complex systems. However, proper care should be taken in achieving a trade-off between the benefit of enhancing the system life and the cost of achieving it subject to other constraints, such as constraints involving weight or volume.

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Received: 18 May 2013 Accepted: 19 November 2013
Published: 4 December 2013

References

1. Barlow, RE, Proschan, F: Statistical Theory of Reliability and Life Testing: Probability Models. To Begin with Publisher, Silver Spring, MD (1981)
2. Mettas, A: Reliability allocation and optimization for complex systems, Proceedings, Annual Reliability and Maintainability Symposium, Los Angeles, California, USA (2000)
3. Huang, HZ: Fuzzy multi-objective optimization decision making of reliability of series system. Microelectron. Reliab. **37**(3), 447–449 (1997)
4. Gen, M, Cheng, R: Optimal design of system reliability using interval programming and genetic algorithms. Comput. Ind. Eng. **31**(1/2), 237–240 (1996)
5. Tillman, FA, Hwang, CL, Kuo, W: Optimization of Systems Reliability. Marcel Dekker, New York (1985)
6. Kuo, W, Prasad, R: An annotated overview of system-reliability optimization. IEEE. T. Reliab. **49**(2), 176–187 (2000)
7. Levitin, G: Redundancy optimization for multi-state system with fixed resource requirements and unreliable sources. IEEE. T. Reliab. **50**(1), 52–59 (2001)
8. Kulturel-Konak, S, Smith, AE, Coit, DW: Efficiently solving the redundancy allocation problem using tabu search. IEEE. Trans. **35**, 515–526 (2003)
9. Shekhalishahi, M, Ebrahimipour, V, Shiri, H, Zaman, H, Jeihoonian, M: A hybrid GA-PSO approach for reliability optimization in redundancy allocation problem. Int. J. Adv. Manuf. Technol. **68**, 317–338 (2013)
10. Cao, D, Murat, A, Chinnam, RB: Efficient exact optimization of multiobjective redundancy allocation problems in series–parallel systems. Reliab. Eng. Syst. Saf. **111**, 154–163 (2013)

doi:10.1186/2195-5468-1-14
Cite this article as: Bhattacharya and Roychowdhury: **Optimal redundancy allocation rule for improving system life.** *Journal of Uncertainty Analysis and Applications* 2013 **1**:14.