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Uncertainty theory based multiple objective mean-entropy-skewness stock portfolio selection model with transaction costs

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Abstract

Purpose: The aim of this paper is to develop a mean-entropy-skewness stock portfolio selection model with transaction costs in an uncertain environment.

Methods: Since entropy is free from reliance on symmetric probability distributions and can be computed from nonmetric data, it is more general than others as a competent measure of risk. In this work, returns of securities are assumed to be uncertain variables, which cannot be estimated by randomness or fuzziness. The model in the uncertain environment is formulated as a nonlinear programming model based on uncertainty theory. Also, some other criteria like short-and long-term returns, dividends, number of assets in the portfolio, and the maximum and minimum allowable capital invested in stocks of any company are considered. Since there is no efficient solution methodology to solve the proposed model, assuming the returns as some special uncertain variables, the original portfolio selection model is transformed into an equivalent deterministic model, which can be solved by any state-of-the-art solution methodology.

Results: The feasibility and effectiveness of the proposed model is verified by a numerical example extracted from Bombay Stock Exchange, India. Returns are considered in the form of trapezoidal uncertain variables. A genetic algorithm is used for simulation.

Conclusions: The efficiency of the portfolio is evaluated by looking for risk contraction on one hand and expected return and skewness augmentation on the other hand. An empirical application has served to illustrate the computational tractability of the approach and the effectiveness of the proposed algorithm.

Keywords: Uncertainty modeling; Mean-entropy-skewness portfolio selection model; Uncertain variables; Trapezoidal uncertain variable; Genetic algorithm

Introduction

The Markowitz [1] formulation of modern portfolio theory has been the most impact-making development in mathematical finance management to date. Since returns are uncertain in nature, the allocation of capital in different risky assets to minimize the risk and to maximize the return is the main concern of it.

In most of the significant works on portfolio selection, the first-order moment of return distribution about the origin, *i.e.*, the mean, quantifies the return, and the second-order moment about the mean, i.e., the variance, quantifies the risk.

Consideration of variance as risk is erroneous as it equally suggests penalties for up and down deviations from the mean. To face this problem, Markowitz [2] recommended semi-variance, a downside risk measure. Another alternative definition of risk is the probability of an adverse outcome [3]. The popular risk measure value at risk [4,5] is in fact an alternative expression of the definition by Roy [3]. Different authors like Philippatos and Wilson [6], Philippatos and Gressis [7], Nawrocki and Harding [8], Simonelli [9], Huang [10], Qin *et al.* [11], and Bhattacharyya *et al.* [12] used entropy as an alternative measure of risk to replace the variance proposed by Markowitz [1]. Uncertainty causes loss and so investors dislike uncertainty. Since entropy is a measure of uncertainty, it is used to measure risk. Entropy is more general than others as an efficient measure of risk because entropy is free from reliance on symmetric distributions and can be computed from nonmetric data.

One of the important theoretical difficulties of these studies is that they assume that asset returns are normally distributed or the utility function is quadratic or that the higher moments are irrelevant to the investors' decision. However, some experimental studies show that portfolio returns are generally not normally distributed. As a result, a natural extension of the mean-variance model is to add the skewness as a factor for consideration in portfolio management. The importance of higher order moments in portfolio selection was suggested by Samuelson [13]. However, considerations of skewness in portfolio selection problem were started by 1990 and were done by Lai [14], Konno and Suzuki [15], Chunhachinda *et al.* [16], Liu *et al.* [17], Prakash *et al.* [18], Briec *et al.* [19], Yu *et al*. [20], Li *et al.* [21], Bhattacharyya *et al.* [22], Bhattacharyya and Kar [23,24], Bhattacharyya [25], and others. Consideration of a mean-entropy-skewness model in portfolio selection problem is introduced by Bhattacharyya *et al.* [12]. They have constructed three portfolio selection models in fuzzy environment using the credibility theory approach.

In most of the abovementioned research works on portfolio selection, the common assumptions are that the investor has enough historical data and that the situation of asset markets in the future can be reflected with certainty by asset data in the past. However, it cannot always be made with certainty. Basically, the usual feature of a financial environment is uncertainty. Mostly, it is realized as risk uncertainty and is modeled by stochastic approaches. However, the term uncertainty has a second aspect-vagueness (imprecision or ambiguity), which can be modeled by fuzzy methodology. In this respect, to tackle the uncertainty in the financial market, stochastic-fuzzy and fuzzy-stochastic methodologies are extensively used in portfolio modeling. Authors like Konno and Suzuki [15], Leon *et al.* [26], Vercher et al. [27], Bhattacharyya *et al.* [12,22], Dey and Bhattacharyya [28], etc. used fuzzy numbers to replace uncertain returns of securities, and they define portfolio selection as a mathematical programming problem in order to select the best alternative. Huang [29] measures portfolio risk by credibility measure and proposed two credibility theory-based mean-variance models. Huang [30] also proposed a mean-semi-variance model for describing the asymmetry of fuzzy returns. She extends the risk definition of variance and chance to a random fuzzy environment and formulates optimization models where security returns are fuzzy random variables.

So, in attempts dealing with portfolio selection problems, randomness and fuzziness are considered as the two basic types of uncertainty contained in security returns. It has become a common practice that when security returns cannot be reflected by historical data, fuzzy variables can be used to show experts' knowledge and estimation of security returns. However, illogicality will come into view if fuzzy variables are used to describe the subjective estimation of security returns. For example, a stock return is considered as a triangular fuzzy variable $\xi = (-0.2, 0.3, 0.7)$. Using possibility theory (or credibility theory), the return is exactly 0.3 with belief degree 1 in possibility measure (or 0.5 in credibility measure). However, this conclusion is unacceptable because the belief degree of exactly 0.3 is almost 0. In addition, the return being exactly 0.3 and not exactly 0.3 has the same belief degree in either possibility measure or credibility measure, which implies that the two events will happen equally likely. This conclusion is quite astonishing and hard to accept.

Again, philosophically, though randomness and fuzziness are two basic types to represent uncertain phenomena, in real life, there are some situations where uncertainty behaves neither randomly nor fuzzily. For example, the occurrence chance of a security price falling in the interval of [100, 110] is 30%, and the occurrence chance of the security price in the interval of [110, 120] is 20%. Then what is the occurrence chance of the security price in the interval of [100, 120]? A survey shows that some people believe that the occurrence chance should be somewhere that is not less than 30% but not greater than 50%. In this case, the security price is neither random nor fuzzy. Recently, Liu [31] proposed an uncertain measure and developed an uncertainty theory, which can be used to handle subjective imprecise quantity. Much research works have been done on the development of uncertainty theory and related theoretical works. Though some considerable amounts of publications have been done in the field of uncertainty theory, not much work has been done in the portfolio selection problem. Huang [32] proposed a mean-risk model for uncertain portfolio selection. Yan [33] found out the deterministic forms of mean-variance portfolio section models corresponding to different special uncertain variables like rectangular uncertain variable, triangular uncertain variable, trapezoidal uncertain variable, and normal uncertain variable. In this study, security returns are considered as uncertain variables, which are characterized by identification functions, and instead of possibility/credibility measure, uncertain measure is used to handle the uncertain events.

Not all the relevant information for an investment decision can be confined in terms of explicit return, risk, and skewness. By capturing additional and alternative decision criteria, a portfolio that is dominated with respect to expected return, skewness, and risk may frame for the shortfall in these three important factors by a very good act on one or several other criteria. As a result, portfolio selection models that consider more criteria than the standard expected return and variance objectives of the Markowitz model have become well liked. Ehrgott *et al.* [34] proposed a model having five criteria, *viz*., short-and long-term return, dividend, ranking, and risk, and used a multicriteria decision making approach to solve the portfolio selection problem. Fang *et al.* [35] proposed a portfolio rebalancing model with transaction costs based on fuzzy decision theory considering three criteria: return, risk, and liquidity.

The main focus of this paper is to propose a mean-entropy-skewness portfolio selection framework with transaction cost having returns in the form of uncertain variables. In addition, it incorporates some useful constraints in the model to make the model more realistic. In addition, this paper provides a real application by using data from

Bombay Stock Exchange (BSE), where we consider returns as trapezoidal uncertain variables.

The rest of the paper is organized as follows. We review the necessary knowledge about uncertainty variables and develop some essential results in the 'Uncertainty theory: related topics' section. In the 'Mean-entropy-skewness model formulation' section, a tri-objective mean-entropy-skewness portfolio selection model is formulated with constraints on short-term and long-term returns, dividends, number of assets in the portfolio, and the maximum and minimum allowable capital invested in stocks of any company. The model is then converted into a single-objective constrained optimization problem with weights over mean, skewness, and entropy. To solve the proposed optimization problem, we provide a genetic algorithm in the 'Genetic algorithm' section. In the 'Case study: Bombay Stock Exchange' section, a case study from Bombay Stock Exchange is done to illustrate the method. The same section also contains a comparative study with other relevant models. Finally, in the last section, some concluding remarks are specified.

Uncertainty theory: related topics

In this paper, the concept of uncertainty theory has been introduced in the field of stock portfolio selection. This section contains only those definitions and theorems on uncertainty theory which are directly used for the formation of this article. The concepts of uncertain measure, uncertain variable, uncertain space, first and second identification functions, rectangular uncertain variable, triangular uncertain variable, exponential uncertain variable, bell-shaped uncertain variable, linear uncertain variable, zigzag uncertain variable, normal uncertain variable, lognormal uncertain variable, and others would be useful to understand the backbone of the article and can be obtained from Liu [31].

Definition 1. A trapezoidal uncertain variable is defined to be the uncertain variable which is fully determined by the four-tuple *(a, b, c, d)* of crisp numbers with $a < b < c < d$, and whose first identification function is

$$\lambda(x) = \begin{cases} \dfrac{x-a}{2(b-a)} & \text{if} \quad a \le x \le b \\ 0.5 & \text{if} \quad b \le x \le c \\ \dfrac{d-x}{2(d-c)} & \text{if} \quad c \le x \le d. \end{cases}$$

Definition 2. The uncertainty distribution $\Phi : \mathbb{R} \to [0, 1]$ of an uncertain variable $\breve{\xi}$ is defined by

$$\Phi(x) = M\{\breve{\xi} \le x\}.$$

Definition 3. An uncertain variable $\breve{\xi}$ is said to have an empirical uncertainty distribution if

$$\phi(x) = \begin{matrix} 0 & \text{if } x < x_i \\ \alpha_i + \dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{(x_{i+1}-x_i)} & \text{if } x_i \le x \le x_{i+1},\ 1 \le i \le n \\ 1 & \text{if } x > x_n \end{matrix}$$

and is denoted by $\varepsilon(x_1, \alpha_1, x_2, \alpha_2, \ldots, x_n, \alpha_n)$, where $x_1 < x_2 < \ldots < x_n$ and $0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le 1$.

Example 1. The trapezoidal uncertain variable $\breve{\xi} = (a, b, c, d)$ follows the empirical uncertain distribution given by

$$\phi(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{2(b-a)} & \text{if } a \le x \le b \\ 0.5 & \text{if } b \le x \le c \\ 0.5 + \dfrac{0.5(x-c)}{d-c} & \text{if } c \le x \le d \\ 1 & \text{if } x > d \end{cases}.$$

Definition 4. The uncertain variables $\breve{\xi}_1, \breve{\xi}_2, \ldots, \breve{\xi}_n$ are said to be independent if

$$M\left\{\bigcap_{i=1}^{n} \breve{\xi}_i\right\} = \min_{1 \le i \le n} M\left\{\breve{\xi}_i \in B_i\right\}$$

for Borel sets $B_1, B_2, \ldots, B_n$ of real numbers. Here M denotes the uncertain measure.

Definition 5. Let $\breve{\xi}$ be an uncertain variable. Then the expected value of $\breve{\xi}$ is given by

$$E(\breve{\xi}) = \int_0^{+\infty} M\{\breve{\xi} \ge r\} dr - \int_{-\infty}^{0} M\{\breve{\xi} \le r\} dr$$

provided that at least one of the two integrals is finite.

Definition 6. Let $\breve{\xi}$ be an uncertain variable. Then the entropy of $\breve{\xi}$ is given by

$$H[\breve{\xi}] = \int_{-\infty}^{\infty} \{-\Phi(x).\ln(\Phi(x)) - (1-\Phi(x))\ln(1-\Phi(x))\} dx.$$

Definition 7. Let $\breve{\xi}$ be an uncertain variable with finite expected value *e*. Then the variance and skewness of $\breve{\xi}$ are respectively given by

$$V[\breve{\xi}] = E\left\{(\breve{\xi}-e)^2\right\},$$

$$S[\breve{\xi}] = E\left\{(\breve{\xi}-e)^3\right\}.$$

Example 2. If $\breve{\xi} = (a, b, c, d)$ is a trapezoidal uncertain variable then

$$E[\breve{\xi}] = \frac{a+b+c+d}{4},$$

$$H[\breve{\xi}] = \frac{b-a+d-c}{2} + (c-b)\ln 2,$$

$$S[\breve{\xi}] = \frac{\{(d-a)^2-(c-b)^2\}\{(d-c)-(d-c)\}}{32}.$$

Theorem 1. *Let $\breve{\xi}_1, \breve{\xi}$ be two independent uncertain variables with finite expected values. Then for any real numbers a and b, we have* $E[a\breve{\xi}_1 + b\breve{\xi}] = aE[\breve{\xi}_1]bE[\breve{\xi}]$.

Theorem 2. *Let $\breve{r}_i = (a_i, b_i, c_i, d_i), (i = 1, 2, \ldots, \ n)$ be n independent uncertain trapezoidal variables and let x_i (i = 1, 2,…, n) be n real variables. Then*

$$\mathrm{E}[\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n] = \frac{1}{4}\sum_{i=1}^{n}(a_i + b_i + c_i + d_i)x_i,$$

$$H[\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n] = \frac{1}{2}\sum_{i=1}^{n}\Big[(b_i - a_i) + (d_i - c_i) + 2(c_i - b_i)\ln 2\Big]x_i,$$

$$S[\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n] = \frac{1}{32}\sum_{i=1}^{n}\Big[\{(d_i - a_i)^2 - (c_i - b_i)^2\}\{(d_i - c_i) - (b_i - a_i)\}\Big]x_i^3.$$

Proof. As $\breve{r}_i = (a_i, b_i, c_i, d_i), (i = 1, 2, \ldots, n)$ are n independent uncertain trapezoidal variables and x_i (*i = 1, 2,…, n*) are n real variables, we have

$$\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots. + \breve{r}_n x_n$$

$$= \sum_{i=1}^{n} \breve{r}_i x_i = \sum_{i=1}^{n}(a_i, b_i, c_i, d_i)x_i$$

$$= \sum_{i=1}^{n}(a_i x_i, b_i x_i, c_i x_i, d_i x_i) = \left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i, \sum_{i=1}^{n} d_i x_i\right).$$

Hence, $\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n$ is a trapezoidal uncertain variable. Combining the above result with the results obtained in Example 2, we are with the theorem.

Mean-entropy-skewness model formulation

In this section, we will first describe the assumptions and notations used in the construction of the paper. Then the objective functions of the models will be constructed in the next subsection. In the third subsection, we will discuss the constraints used in our portfolio selection model. The fourth subsection will include three different mathematical models for different situations.

Assumptions and notations

Let us consider a financial market with n risky assets offering uncertain returns. An investor allocates his wealth among these risky assets.

For the ith risky asset *(i* = 1, 2,…, *n)*, let us use the following notations:

x_i = portion of the total capital invested in ith security

$\breve{p}_i$ = uncertain variable representing the closing price of the ith security at present

$\breve{p}_i'$ = uncertain variable representing the estimated closing price of the ith security in the next year

d_i = the estimated dividends in the next year

$\breve{r}_i = \frac{\breve{p}_i' + d_i - \breve{p}_i}{\breve{p}_i}$ = uncertain variable representing the return of the ith security

$R_i^{(12)}$ = the average 12 month performance

$R_i^{(36)}$ = the average 36 month performance

k_i = the constant transaction cost per change in a proportion, $k_i \geq 0$

$$y_i = \begin{cases} 1 & \text{if the } i\text{th asset is contained in the portfolio} \\ 0 & \text{if the } i\text{th asset is not contained in the portfolio.} \end{cases}$$

Formulation of objective functions

It is impossible to predict future returns of stocks in any budding security market. The arithmetic mean of historical data is in general considered as the expected return of securities, which yield us a crisp value. However, for this technique, two main problems need to be solved. Firstly, if historical data for a long period are considered, the influence of earlier historical data is the same as that of recent data, whereas recent data of a security is more important than the earlier historical data. Secondly, if the historical data of a security are not adequate, due to the lack of information, the estimations of the statistical parameters are not adequate. For these reasons, the expected return of a security is considered here as an uncertain variable instead of the crisp arithmetic mean of historical data. Similarly, in an uncertain environment, the risk (entropy) and skewness cannot be predicted exactly. Therefore, the entropy and skewness are also considered here as uncertain variables.

Let us consider the transaction cost c_i to be a V-shaped function of the difference between a given portfolio $x^0 = \left(x_1^0, x_2^0, \ldots, x_n^0\right)$ and a new portfolio $x = (x_1, x_2, \ldots, x_n)$ and is incorporated explicitly into the portfolio return. Thus, the transaction cost of ith risky asset can be expressed as

$$c_i = k_i\left|x_i - x_i^0\right|, i = 1, 2, 3, \ldots, n.$$

Hence the total transaction cost is

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} k_i\left|x_i - x_i^0\right|.$$

The expected return of portfolio $x = (x_1, x_2, \ldots, x_n)$ with transaction cost is thus given by

$$\mathrm{Re}(x) = E[(\breve{r}_1 - c_1)x_1 + (\breve{r}_2 - c_2)x_2 + \ldots + (\breve{r}_n - c_n)x_n].$$

The entropy of portfolio $x = (x_1, x_2, \ldots, x_n)$ is given by

$$\mathrm{En}(x) = H\,[\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n].$$

The skewness of portfolio $x = (x_1, x_2, \ldots, x_n)$ is given by

$$\mathrm{Sk}(x) = S\,[\breve{r}_1 x_1 + \breve{r}_2 x_2 + \ldots + \breve{r}_n x_n].$$

We consider the portfolio selection problem as a tri-objective optimization problem. As discussed earlier, the objectives we consider are

$$\begin{cases} \text{Max} & \mathrm{Re}(x) \\ \text{Min} & \mathrm{En}(x) \\ \text{Max} & \mathrm{Sk}(x). \end{cases}$$

Construction of the constraints

For the portfolio $x = (x_1, x_2,..., x_n)$, the expected short-term return is expressed as

$$R_{st}(x) = \sum_{i=1}^{n} R_i^{(12)} x_i - \sum_{i=1}^{n} k_i \left| x_i - x_i^0 \right|.$$

For the portfolio $x = (x_1, x_2,..., x_n)$, the expected long-term return is expressed as

$$R_{lt}(x) = \sum_{i=1}^{n} R_i^{(36)} x_i - \sum_{i=1}^{n} k_i \left| x_i - x_i^0 \right|.$$

Since investors plan their asset allocation on short-term, long-term, or both cases, they should prefer a portfolio having at least a minimum short-term, long-term, or both types of return. For that reason, we consider the following two constraints:

$$R_{st}(x) \geq \varsigma,$$
$$R_{lt}(x) \geq \tau,$$

where ς and τ will be allocated by the investor.

Dividend is the payment made by a company to its shareholders. It is the portion of corporate profits paid out to the investors. For the portfolio $x = (x_1, x_2,..., x_n)$, the annual dividend is expressed as

$$D(x) = \sum_{i=1}^{n} d_i x_i.$$

Clearly, investors would like to have a portfolio that yields them a high dividend. Keeping in mind this fact, we propose the following constraint:

$$D(x) \geq d,$$

where d will be allocated by the investor.

The well-known capital budget constraint on the assets is presented by

$$\sum_{i=1}^{n} x_i = 1.$$

The maximum and minimum fractions of the capital budget being allocated to each of the assets in the portfolio depend upon factors like price relative to the asset in comparison with the average of the price of all the assets in the chosen portfolio, minimal lot size that can be traded in the market, the past performance of the price of the asset, information available about the issuer of the asset, trends in the business of which it is a division, etc. That is, an investor will have to look upon a host of the basics affecting the commerce. Different investors having different views may allocate the same overall capital budget differently.

Let the maximum fraction of the capital that can be invested in a single asset i be M_i. Then

$$x_i \leq M_i y_i \forall i = 1, 2, ..., n.$$

Let the minimum fraction of the capital that can be invested in a single asset i be m_i. Then

$$x_i \geq m_i y_i \;\; \forall i = 1, 2, \ldots, n.$$

The investor would like to pick up the assets among all the assets in a given set that in his subjective estimate are likely to yield the greatest performance. Thus it is not necessary that all the assets in the given set may configure in the portfolio. Investors can thus consider the number of assets they can effectively handle in a portfolio.

Let the number of assets held in a portfolio be k. Then

$$\sum_{i=1}^{n} y_i = k.$$

As no short selling is considered, we have

$$x_i \geq 0 \;\; \forall i = 1, 2, \ldots, n.$$

If X is the set of feasible portfolios, then we have,

$$X = \{x = \{x_1, \; x_2, \ldots, \; x_n\} \text{ such that } R_{\mathrm{st}}(x) \geq \varsigma, \; R_{\mathrm{lt}}(x) \geq \tau, \; D(x) \geq d, \; x_i \leq M_i y_i, \; x_i \geq m_i y_i, \; \sum_{i=1}^{n} y_i = k, \; \sum_{i=1}^{n} x_i = 1, \; x_i \geq 0\}. \tag{1}$$

Weighted portfolio selection model formulation

The portfolio selection model is thus formulated as

$$\begin{cases} \text{Max} & \text{Re} \\ \text{Min} & \text{En} \\ \text{Max} & \text{Sk} \\ x \in X. & \end{cases} \tag{2}$$

To convert the above tri-objective optimization problem into a preference-based single-objective optimization problem, let us consider three single-objective optimization problems optimizing separately the three objectives of the model subject to the constraints of the problem. The optimum values as well as the values of the remaining objective functions in each of the three cases are calculated. Considering all the three problems, let the minimum values of the three objectives be $\text{Re}^{\min}$, $\text{En}^{\min}$, and $\text{Sk}^{\min}$, respectively. Also, let the maximum values of the three objectives be $\text{Re}^{\max}$, $\text{En}^{\max}$, and $\text{Sk}^{\max}$, respectively. Then the above tri-objective portfolio selection model is transformed into the following model:

$$\begin{cases} \text{Max} \left[w_1 \dfrac{\text{Re}(x) - \text{Re}^{\min}}{\text{Re}^{\max} - \text{Re}^{\min}} + w_2 \dfrac{\text{En}^{\max} - \text{En}(x)}{\text{En}^{\max} - \text{En}^{\min}} + w_3 \dfrac{\text{Sk}(x) - \text{Sk}^{\min}}{\text{Sk}^{\max} - \text{Sk}^{\min}} \right] \\ \text{subject to} \\ \quad x \in X \\ \quad w_1 + w_2 + w_3 = 1 \end{cases} \tag{3}$$

where w_1, w_2, and w_3 are weights or preferences to the objectives *Re(x)*, *En(x)*, and *Sk(x)*, respectively. w_1, w_2, and w_3 will be allocated by the investor.

Genetic algorithm

After development of the genetic algorithm (GA) by Holland in 1975, it has been extensively used/modified to solve complex decision making problems in different fields of science and technology. A GA normally starts with a set of potential solutions (called initial population) of the decision making problem under consideration. Individual solutions are called chromosome. Crossover and mutation operations happen among the potential solutions to get a new set of solutions, and the process continues until terminating conditions are encountered. The following functions and values are adopted in the proposed GA to solve the problem [36]. The different parameters on which this GA depends are the number of generation (*MAXGEN*), population size (*POPSIZE*), probability of crossover (*PCROS*), and probability of mutation (*PMUTE*).

Chromosome representation

An important issue in applying a GA is to design an appropriate chromosome representation of solutions of the problem together with genetic operators. Traditional binary vectors used to represent the chromosome are not effective in many nonlinear problems. Since the proposed model is highly nonlinear, hence, to overcome the difficulty, a real-number representation is used. In this representation, each chromosome V_i is a string of n number of genes G_{ij} (i = 1, 2,…, POPSIZE, j = 1, 2,…, n) where these n number of genes respectively denote n number of decision variables x_j.

Initial population production

For each chromosome V_i, every gene G_{ij} is randomly generated between its boundary (LB_j, UB_j) where LB_j and UB_j are the lower and upper bounds of the variables x_j (j = 1, 2,…, n and i = 1, 2,…, POPSIZE), respectively.

Evaluation

Evaluation function plays the same role in GA as that the environment plays in natural evolution. Now, evaluation function (EVAL) for chromosome V_i is equivalent to the objective function $f(x_1, x_2,\ldots, x_n)$. The following are the steps of evaluation:

1. Find $\mathrm{EVAL}(V_i) = f(x_1, x_2,\ldots, x_n)$, where the genes G_{ij} represent the decision variable x_j, j = 1, 2,…, n and f is the objective function.
2. Find total fitness of the population: $F = \sum_{i=1}^{\mathrm{POPSIZE}} \mathrm{EVAL}(V_i)$.
3. The probability p_i of selection for each chromosome V_i is determined by the formula $p_i = \frac{1}{F}\mathrm{EVAL}(V_i)$.
4. Calculate the cumulative probability Y_i of selection for each chromosome V_i by the formula $Y_i = \sum_{j=1}^{i} p_i$.

Selection

The selection scheme in GA determines which solutions in the current population are to be selected for recombination. Many selection schemes, such as stochastic random sampling roulette wheel selection, have been proposed for various problems. In this paper, we adopt the roulette wheel selection process. This roulette wheel selection process is based on spinning the roulette wheel *POPSIZE* times each time we select a single chromosome for the new population in the following way:

(a) Generate a random (float) number r between 0 and 1.
(b) If $r < Y_1$, then the first chromosome is V_1; otherwise, select the ith chromosome V_i (2 ≤ i ≤ POPSIZE) such that $Y_{i-1} \leq r < Y_i$.

Crossover

A crossover operator is mainly responsible for the search of new strings. Crossover operates on two parent solutions at a time and generates offspring solutions by recombining both parent solution features. After selection of chromosomes for new population, the crossover operator is applied. Here, the arithmetic crossover operation is used. It is defined as a linear combination of two consecutive selected chromosomes V_m and V_n, and resulting offspring's V'_m and V'_n are calculated as

$$V'_m = cV_m + (1-c)V_n,$$
$$V'_n = cV_n + (1-c)V_m,$$

where c is a random number between 0 and 1.

Mutation

A mutation operator is used to prevent the search process from converging to local optima rapidly. It is applied to each single chromosome V_i. The selection of a chromosome for mutation is performed in the following way:

1. Set $i \leftarrow 1$.
2. Generate a random number u from the range [0, 1].
3. If $u <$ PMUTE, then we select the chromosome V_i.
4. Set $i \leftarrow i + 1$.
5. If $i \leq$ POPSIZE, then go to step 2. Then the particular gene G_{ij} of the chromosome V_i selected by the abovementioned steps is randomly selected. In this problem, the mutation is defined as G_{ij}^{mut} random number from the range (LB_j, UB_j).

Termination

If the number of iteration is less than or equal to MAXGEN, then the process goes on; otherwise, it terminates.

Proposed GA procedure

Start
{
$t \leftarrow 0$
while (all constraints are not satisfied)
{
initialize Population (t)
}
evaluate Population (t)
while(not terminate - condition)
{
$t \leftarrow t + 1$
select Population (t) from Population ($t - 1$)
crossover and mutate Population (t)

evaluate Population (t)
}
print optimum result
}.

Case study: Bombay Stock Exchange (BSE)

Bombay Stock Exchange is the oldest stock exchange in Asia with a rich heritage of over 133 years of existence. What is now popularly known as BSE was established as 'The Native Share & Stock Brokers' Association' in 1875. It is the first stock exchange in India which obtained permanent recognition (in 1956) from the Government of India under the Securities Contracts (Regulation) Act (SCRA) 1956. With demutualization, the stock exchange has two of world's prominent exchanges, Deutsche Borse and Singapore Exchange, as its strategic partners. Today, BSE is the world's number one exchange in terms of the number of listed companies and the world's fifth in handling of transactions through its electronic trading system. The companies listed on BSE command a total market capitalization of US$1.06 trillion as of July 2009.

The BSE index, SENSEX, is India's first and most popular stock market benchmark index. SENSEX is tracked worldwide. It constitutes 30 stocks representing 12 major sectors. It is constructed on a 'free-float' methodology, and is sensitive to market movements and market realities. Apart from SENSEX, BSE offers 23 indices, including 13 sectoral indices.

Case study

We have taken monthly share price data for 60 months (March 2003 to February 2008) of just five companies which are included in the BSE index. Though any finite number of stocks can be considered, we have taken only five stocks to reduce the complexity of representation.

The Table 1 shows the stocks along with their returns in the form of trapezoidal uncertain numbers, the average short-term returns, the average long-term returns, and the dividends. We also have k_i = 0.001. We consider, x_i^0 = 0 for i = 1, 2, 3, 4, 5.

Example

With respect to the above data, we consider the following tri-objective portfolio selection model:

Table 1 Stocks information

Stock	Return ($\tilde{r}_i$)	Short-term returns ($R_i^{(12)}$)	Long-term return ($R_i^{(36)}$)	Dividends (d_i, %)
Reliance energy	(−0.008, 0.020, 0.042,0.067)	0.0324	0.031	63
L&T	(−0.003, 0.029, 0.057, 0.087)	0.0524	0.044	85
Bhel	(−0.002,0.021, 0.051, 0.083)	0.0510	0.037	125
Tata steel	(0.009, 0.023, 0.038, 0.052)	0.0307	0.032	155
SBI	(−0.010, 0.022, 0.045, 0.079)	0.0387	0.035	140

Table 2 Solution

	Case 1	Case 2	Case 3	Case 4
w_1	1/3	0.6	0.2	0.2
w_2	1/3	0.2	0.6	0.2
w_3	1/3	0.2	0.2	0.6
Re(x)	0.03830	0.03878251	0.03787975	0.0389781
En(x)	0.0667776	0.06823548	0.06466012	0.06811745
Sk(x)	0.000000321	0.000002439285	0.0000002561417	0.000001260730
Dividend (%)	110.2186	113.1965	112.4341	110.2661
x_1	0	0	0	0
x_2	0.4922520	0.3969638	0.4552814	0.4865646
x_3	0.1805016	0.3313681	0.3565393	0
x_4	0	0	0.1881792	0.1981891
x_5	0.3272464	0.2716681	0	0.3152462

$$\begin{cases} \text{Max } \operatorname{Re}(x) \\ \text{Min } \operatorname{En}(x) \\ \text{Max } \operatorname{Sk}(x) \\ \text{subject to} \\ R_{st}(x) \geq 0.039, \; R_{lt}(x) \geq 0.038, \; D(x) \geq 1.1, \\ x = (x_1, \; x_2, \; x_3, \; x_4, \; x_5), y_i \in \{0,1\} \\ x_i \leq 0.6y_i, x_i \geq 0.1y_i, \; \sum_{i=1}^{5} x_i = 1, \; \sum_{i=1}^{5} y_i = 3. \end{cases} \tag{4}$$

Solution

To solve the above example, the GA is used with the parameters POPSIZE = 50, PCROS = 0.2, PMUTE = 0.2, *and* MAXGEN = 100. A real-number presentation is used here. In this representation, each chromosome x is a string of m (here, $m = 5$) number of genes; these represent decision variables. For each chromosome x, every gene (here, x_1, x_2, x_3, x_4, x_5) is randomly generated between its boundaries until it is feasible. In this

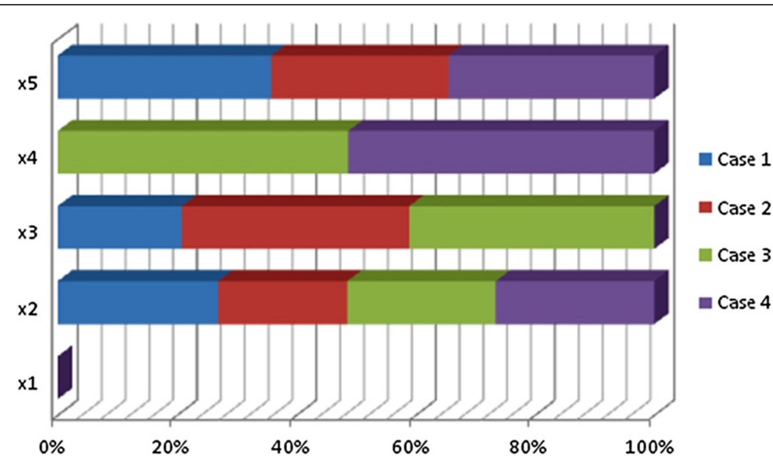

Figure 1 Portfolios.

Table 3 Individual mean, entropy, and skewness of the stocks

Stock	Return	Entropy	Skewness
Reliance energy	0.03025	0.0417492	-4.8197×10^{-7}
L&T	0.04250	0.0504081	-1.4632×10^{-5}
Bhel	0.03825	0.0384751	2.0318×10^{-6}
Tata steel	0.03050	0.0243972	0
SBI	0.0340	0.0489424	4.62×10^{-7}

problem, arithmetic crossover and random mutation are applied to generate new offsprings.

As discussed in the 'Weighted portfolio selection model formulation' section, optimizing the three single objectives Re(x), En(x), and Sk(x) separately subject to the constraints in (4), we obtain the minimum and maximum values of the objectives with the same parameters. In each case, only the best solution is considered.

With reference to model (3), the problem (4) is transformed into the following model:

$$\begin{cases} \text{Max} \left[w_1 \dfrac{\text{Re}(x) - 0.0346375}{0.03957812 - 0.0346375} + w_2 \dfrac{0.06896848 - \text{En}(x)}{0.06896848 - 0.05826382} + w_3 \dfrac{\text{Sk}(x) - (-0.000000199502)}{0.000000384282 - (-0.000000199502)} \right] \\ \text{subject to} \\ R_{\text{st}}(x) \geq 0.039, \;\; R_{\text{lt}}(x) \geq 0.038, \;\; D(x) \geq 1.1, \;\; x = (x_1, \; x_2, \; x_3, \; x_4, \; x_5) \\ \sum_{i=1}^{5} x_i = 1, \;\; x_i \leq 0.6 \; y_i, \;\; x_i \geq 0.1 y_i, \;\; \sum_{i=1}^{5} y_i = 3, \;\; y_i \in \{0, 1\}, w_1 + w_2 + w_3 = 1. \end{cases} \tag{5}$$

For different preassigned values of w_1, w_2, and w_3, the above problem is solved. We have considered only the best solutions. The solutions obtained are shown in Table 2.

In case 1, where an investor gives same importance to all the three objectives, the portfolio states that the investor should invest 45%, 45%, and 10% of the money to the second, third, and fourth stocks, respectively. In case 3, where the importance is given towards minimization of risk, the investor should invest 39.6%, 18.8%, and 41.6% of the total money to the second, fourth, and fifth stocks, respectively. Similarly, we can explain the other two cases.

In case 2, where more importance is given to return, the investor gets a return of 0.03957813 which is higher than that of the other three cases {0.03893750, 0.03586275, and 0.03830000}. In case 3, where more importance is given to risk, the investors' risk (0.09320499) is lower than in all other cases {0.09551172, 0.09846223, and 0.09487761}. Similarly, in case 4, we get the best result for skewness. In case 1, where equal importance is given to all objectives, the outputs are intermediate. We represent the portfolios obtained in cases 1, 2, 3, and 4 graphically in Figure 1.

Table 4 Solution of the model in (5.4.1)

x_1	x_2	x_3	x_4	x_5	E
0.000000	0.3840023	0.5159977	0.000000	0.1000000	0.03945701

Table 5 Solution of the model of Ning et al

x_1	x_2	x_3	x_4	x_5	E
0.000000	0.4125000	0.4875000	0.000000	0.1000000	0.03957813

Some questions may arise on the appropriateness of the portfolios obtained in Table 2 under different circumstances (cases 1, 2, 3 and 4). For example, question may arise on the absence of reliance energy in all the obtained portfolios. To explain that, the individual mean, entropy and skewness of the stocks are calculated by Example 2 and are shown in Table 3. It is seen that reliance energy has the lowest rerun among the five stocks. It is also possessing negative skewness. So, the absences of reliance energy on the portfolios in cases 1, 2 and 4 are obvious. In case 3, where more importance is given to entropy, the selected portfolio contains L&T, Tata steel and Bhel. Tata steel and Bhel are the two stocks with lowest risks. Again, though L&T has a higher risk, it also has very high return. So, the portfolio in case 3 is not compromising too much towards entropy and is maintaining the characteristic of multiobjective optimization. This is also to note that if the constraint $x_i \geq 0.1y_i$ is not considered, then some of the portfolios would contain non-zero x_1.

Comparative study

We compare the results in Table 2 with other relevant literature to demonstrate how the results from the proposed technique compare with the literatures of uncertainty theory in the portfolio selection problem. Thus, the models in [33], [37], and [38] which apply uncertainty theory in portfolio selection are considered with the same data set as that in Table 1. We also used the following set of constraints (X) for each case:

We also used the following set of constraints (X) for each case:

$$X = \{R_{st}(x) \geq 0.039,\ \ R_{lt}(x) \geq 0.038,\ \ D(x) \geq 1.1,\ \ x_i \leq 0.6y_i,\ \ x_i \geq 0.1y_i,\ \ \sum_{i=1}^{5} x_i = 1,\ \ \sum_{i=1}^{5} y_i = 3\}.$$

Model of Yan

We considered the following model [33]:

$$\left.\begin{array}{l} \text{Maximize } E[\breve{r}_1x_1 + \breve{r}_2x_2 + \breve{r}_3x_3 + \breve{r}_4x_4 + \breve{r}_5x_5] \\ \text{Subject to the constraints} \\ V[\breve{r}_1x_1 + \breve{r}_2x_2 + \breve{r}_3x_3 + \breve{r}_4x_4 + \breve{r}_5x_5] \leq 0.002 \\ x \in X \end{array}\right\}.$$

Here E stands for mean (return) and V stands for variance (risk). The solution is shown in Table 4.

Table 6 Solution of the model of Liu and Qin

x_1	x_2	x_3	x_4	x_5	E
0.000000	0.2333333	0.6000000	0.000000	0.1666667	0.03853333

Model of Ning et al

We considered the following model [37]:

$$\left.\begin{array}{l}\text{Maximize } E[\breve{r}_1x_1+\breve{r}_2x_2+\breve{r}_3x_3+\breve{r}_4x_4+\breve{r}_5x_5]\\ \text{Subject to the constraints}\\ \text{TVaR}[\breve{r}_1x_1+\breve{r}_2x_2+\breve{r}_3x_3+\breve{r}_4x_4+\breve{r}_5x_5]\le -0.008\\ x\in X\end{array}\right\}.$$

Here E stands for mean (return) and $TvaR$ stands for tail value at risk. The solution is shown in Table 5.

Model of Liu and Qin

We considered the following model [38]:

$$\left.\begin{array}{l}\text{Maximize } E[\breve{r}_1x_1+\breve{r}_2x_2+\breve{r}_3x_3+\breve{r}_4x_4+\breve{r}_5x_5]\\ \text{Subject to the constraints}\\ \text{SAD}[\breve{r}_1x_1+\breve{r}_2x_2+\breve{r}_3x_3+\breve{r}_4x_4+\breve{r}_5x_5]\le 0.0144\\ x\in X\end{array}\right\}.$$

Here E stands for mean (return) and SAD stands for semi-absolute deviation (risk). The solution is shown in Table 6.

In the discussions done in the first and second sections, we see that using entropy as a measure of risk/uncertainty is analytically better than the other conventional measures. Again, if we compare Tables 4, 5, and 6 with Table 2, we see that the performance of the proposed model is clearly at par or better than the established models.

Conclusions

This paper has introduced a new framework of mean-entropy-skewness portfolio selection problem with transaction cost under the constrains on short-and long-term returns with transaction costs, dividends, number of assets in the portfolio, and the maximum and minimum allowable capital invested in stocks. Uncertainties of future return of stocks are characterized by uncertain variables. The efficiency of the portfolios is evaluated by looking for risk contraction on one hand and expected return and skewness augmentation on the other hand. An empirical application has served to illustrate the computational tractability of the approach and the effectiveness of the proposed algorithm. A comparative study with other relevant literatures proves the usefulness of the proposed model. In addition to the GA, some other meta-heuristic algorithms such as tabu search, simulated annealing, ant colony optimization, and particle swam optimization may be employed to solve the nonlinear programming problem.

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