Verma and Sharma *Journal of Uncertainty Analysis and Applications* 2014, **2**:10
http://www.juaa-journal.com/content/2/1/10

Journal of Uncertainty Analysis and Applications
a SpringerOpen Journal

RESEARCH Open Access

Trapezoid fuzzy linguistic prioritized weighted average operators and their application to multiple attribute group decision-making

Rajkumar Verma* and Bhu Dev Sharma

* Correspondence: rkver83@gmail.com
Department of Mathematics, Jaypee Institute of Information Technology (Deemed University), Noida–201307, Uttar Pradesh, India

Abstract

The prioritized weighted average (PWA) operator was originally introduced by Yager. The prominent characteristic of the PWA operator is that it takes into account prioritization among attributes and decision makers. Motivated by the idea of PWA operator, we develop some prioritized weighted aggregation operators for aggregating trapezoid fuzzy linguistic information. The properties of the new aggregation operators are studied in detail. Furthermore, based on the proposed operators, some approaches to deal with multiple attribute group decision-making problems under trapezoid fuzzy linguistic environments are developed. Finally, a practical example is provided to illustrate the multiple attribute group decision-making process.

Keywords: Multiple attribute group decision-making (MAGDM); Trapezoid fuzzy linguistic variables; Prioritized weighted average operator; Trapezoid fuzzy linguistic prioritized weighted average operator; Trapezoid fuzzy linguistic prioritized weighted geometric operator; Trapezoid fuzzy linguistic prioritized weighted harmonic average operator

Introduction

In the process of multiple attribute decision-making, information aggregation is an essential process of gathering relevant information from various sources. In the literature, a wide range of aggregation operators are found for aggregating the data information [1-4]. The ordered weighted averaging (OWA) operator, introduced by Yager [5], is a well-known aggregation operator that provides a parameterized family of aggregation operators, including the maximum, the minimum, and the average. Since its appearance, the OWA operator has received increasing attention from many authors and it has been applied across many fields [6-28]. Chiclana et al. [8] and Xu and Da [19] introduced the ordered weighted geometric (OWG) operators, which are based on the OWA operator and on the geometric mean. A further interesting extension of the OWA operator is the generalized OWA (GOWA) operator [26] that uses generalized means [29] in the OWA operator.

However, in some situations, the input arguments take the form of linguistic variables, rather than being real numbers because of time pressure, lack of knowledge, and people's limited expertise related with problem domain. Bordonga et al. [6] utilized the OWA operator to solve the group decision-making problem in linguistic context. Herrera and Martínez [30] established a linguistic 2-tuple computational model for dealing with linguistic information. To aggregate uncertain linguistic information, Xu [31] proposed the uncertain linguistic weighted averaging (ULWA) operator, the uncertain linguistic ordered weighted averaging (ULOWA) operator, and the uncertain linguistic hybrid averaging (ULHA) operator. Xu [22] introduced some uncertain linguistic geometric mean operators including the uncertain linguistic geometric mean (ULGM), the uncertain linguistic weighted geometric mean (ULWGM) operator, the uncertain linguistic ordered weighted geometric mean (ULOWGM) operator, and the induced uncertain linguistic ordered weighted geometric mean (IULOWGM) operator and developed an approach to group decision-making with uncertain multiplicative linguistic relation. Wei [32] defined the uncertain linguistic hybrid geometric mean (ULHGM) and applied it to the group decision-making. Further, Xu [33] proposed some aggregation operators for aggregating triangular fuzzy linguistic information such as the fuzzy linguistic averaging (FLA) operator, the fuzzy linguistic weighted averaging (FLWA) operator, the fuzzy linguistic ordered weighted averaging (FLOWA) operator, and the induced fuzzy linguistic ordered weighted averaging (IFLOWA) operator. The trapezoid fuzzy linguistic variable (TFLV), introduced by Xu [34], generalizes the linguistic variable, the uncertain linguistic variable, and the triangular fuzzy linguistic variable, and research on aggregation operators under trapezoid fuzzy linguistic environment is very significant. Xu [34] and Liang and Chen [35] proposed the trapezoid fuzzy linguistic weighted averaging (TFLWA) operator and applied it to multiple attribute decision-making problems. Wei and Yi [36] introduced the trapezoid fuzzy linguistic weighted geometric mean (TFLWGM) operator and developed an approach to group decision-making with trapezoid fuzzy linguistic information. Liu and Su [37] introduced the trapezoid fuzzy linguistic ordered weighted averaging (TFLOWA) operator and the trapezoid fuzzy linguistic hybrid ordered weighted averaging (TFLHOWA) operator. Further, Liu and Su [38] developed some trapezoid fuzzy linguistic harmonic averaging operators such as the trapezoid fuzzy linguistic weighted harmonic averaging (TFLWHA) operator, the trapezoid fuzzy linguistic ordered weighted harmonic averaging (TFLOWHA) operator, and the trapezoid fuzzy linguistic hybrid harmonic averaging (TFLHHA) operator, and then studied some desirable properties of these operators. Based on the idea of Bonferroni mean [39], Liu and Jin [40] proposed some Bonferroni mean operators such as the trapezoid fuzzy linguistic Bonferroni mean (TFLBM), the trapezoid fuzzy linguistic weighted Bonferroni mean (TFLWBM), the trapezoid fuzzy linguistic Bonferroni OWA (TFLBOWA), and the trapezoid fuzzy linguistic weighted Bonferroni OWA (TFLWBOWA) for aggregating trapezoid fuzzy linguistic correlative information. Recently, on the basis of the idea of the generalized mean [29], Liu and Wu [41] proposed some generalized trapezoid fuzzy linguistic aggregation operators and found their application in multiple attribute group decision-making.

Trapezoid fuzzy linguistic variables are very useful tools to deal with uncertain or fuzzy information. In the last couple of years, many multiple attribute group decision-making theories and methods have been proposed under trapezoid fuzzy linguistic environments with the assumption that the attributes and the decision makers are at the same priority levels. However, in the real life multiple attribute group decision-making problems, attributes and decision makers have different priority levels in general. To overcome this issue, motivated by the idea of prioritized weighted aggregation operators [42,43], in this paper, we propose some trapezoid fuzzy linguistic prioritized weighted aggregation operators such as the trapezoid fuzzy linguistic prioritized weighted average (TFLPWA) operator, the trapezoid fuzzy linguistic prioritized weighted geometric (TLLPWG) operator, and the trapezoid fuzzy linguistic prioritized weighted harmonic (TFLPWH) operator. A prominent characteristic of these proposed operators is that they take into account the prioritization among the attributes and decision makers. Further, we have utilized these operators to develop some approaches to solve multiple attribute group decision-making problems under trapezoid fuzzy linguistic environments.

The paper is organized as follows. In the 'Preliminaries' section, some basic concepts related to trapezoid fuzzy linguistic variables and prioritized weighted average operator are briefly given. In the 'Trapezoid fuzzy linguistic prioritized weighted aggregation operators' section, we introduce some trapezoid fuzzy linguistic prioritized weighted aggregation operators: the trapezoid fuzzy linguistic prioritized weighted average (TFLPWA) operator, the trapezoid fuzzy linguistic prioritized weighted geometric (TFLPWG) operator, and the trapezoid fuzzy linguistic prioritized weighted harmonic average (TFLPWHA) operator. Some properties of proposed operators are also studied here. In the 'An approach to multiple attribute group decision-making with trapezoid fuzzy uncertain linguistic information' section, we have applied these operators to develop some decision models for solving trapezoid fuzzy linguistic multiple attribute group decision-making problems in which the attributes and decision makers are in different priority levels. In the 'Numerical example' section, a numerical example is presented to illustrate the proposed approach to multiple attribute group decision-making with trapezoid fuzzy linguistic information. Our conclusions are presented in the 'Conclusions' section.

Preliminaries

In this section, we briefly review some basic concepts related to trapezoid fuzzy linguistic variables and prioritized weighted average operator, which will be needed in the following analysis.

Let $S = \{s_i | i = 1, 2,\ldots,t\}$ be a discrete linguistic term set with odd cardinality. Any label, s_i, represents a possible value for a linguistic variable, and it must have the following characteristics [31]:

(i) The set is ordered: $s_i \geq s_j$ if $i \geq j$.
(ii) There is the negation operator: $\text{neg}(s_i) = s_j$ such that $j = t - i$.
(iii) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$.
(iv) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

For example, S can be defined as

$$S = \{ s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{slightly poor}, s_5 = \text{fair}$$
$$s_6 = \text{slightly good}, s_7 = \text{good},\ \ s_8 = \text{very good}, s_9 = \text{extremely good}\}.$$

Further, we extend the discrete term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha | s_1 \leq s_\alpha \leq s_t, \alpha \in [1, t]\}$. If $s_\alpha \in S$, then, we call s_α an original linguistic term, otherwise, we call s_α the virtual linguistic term [22]. In general, the decision makers use the original linguistic term to evaluate alternatives, and the virtual linguistic terms can only appear in calculation [31].

Definition 1. Distance between two linguistic variables [33]: Let s_α and s_β be two linguistic variables, then the distance between s_α and s_β is defined as follows:

$$d(s_\alpha, s_\beta) = |\alpha - \beta|. \tag{1}$$

In some situations, however, the decision makers (DMs) may provide fuzzy linguistic information because of time pressure, lack of knowledge, and their limited expertise related with the problem domain. To handle such type of cases, Xu [34] defined the trapezoid fuzzy linguistic variable and introduced some of the operational laws on them.

Definition 2. Trapezoid fuzzy linguistic variable [34]: Let $\tilde{s} = [s_\alpha, s_\beta, s_\gamma, s_\eta]$, where s_α, s_β, s_γ, $s_\eta \in S$, and the subscripts α, β, γ, and η are non-decreasing numbers and s_β and s_γ indicate the interval in which the membership value is 1, with s_α and s_η indicating the lower and upper values of $\tilde{s}$, respectively. Then, $\tilde{s}$ is called the trapezoid fuzzy linguistic variable, which is characterized by the following membership function (see Figure 1):

$$\mu_{\tilde{s}}(\theta) = \begin{cases} 0, & s_0 \le s_\theta \le s_\alpha, \\ \dfrac{d(s_\theta, s_\alpha)}{d(s_\beta, s_\alpha)}, & s_\alpha \le s_\theta \le s_\beta, \\ 1 & s_\beta \le s_\theta \le s_\gamma, \\ \dfrac{d(s_\theta, s_\eta)}{d(s_\gamma, s_\eta)}, & s_\gamma \le s_\theta \le s_\eta, \\ 0, & s_\eta \le s_\theta \le s_q. \end{cases} \tag{2}$$

Especially, if any two of α, β, γ, and η are equal, then $\tilde{s}$ is reduced to a triangular fuzzy linguistic variable [33], and if any three of α, β, γ, and η are equal, then $\tilde{s}$ is reduced to an uncertain linguistic variable [31].

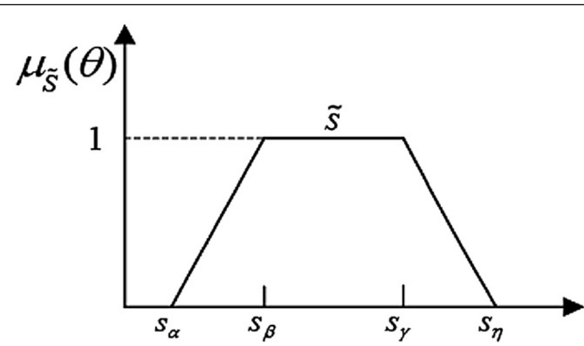

Figure 1 A trapezoid fuzzy linguistic variable.

Definition 3. Arithmetical operations on trapezoid fuzzy linguistic variables [34,37,41]: Let $\tilde{s} = [s_\alpha, s_\beta, s_\gamma, s_\eta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1}, s_{\eta_1}]$, and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2}, s_{\eta_2}]$ be three trapezoid fuzzy linguistic variables, then some arithmetical operations are defined as follows:

(i) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1}, s_{\eta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2}, s_{\eta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}, s_{\gamma_1+\gamma_2}, s_{\eta_1+\eta_2}]$,

(ii) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1}, s_{\eta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2}, s_{\eta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}, s_{\gamma_1\gamma_2}, s_{\eta_1\eta_2}]$,

(iii) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta, s_\gamma, s_\eta] = [s_{\lambda\alpha}, s_{\lambda\beta}, s_{\lambda\gamma}, s_{\lambda\eta}], \quad \lambda \geq 0$

(iv) $\tilde{s}^\lambda = [s_\alpha, s_\beta, s_\gamma, s_\eta]^\lambda = \left[s_{\alpha^\lambda}, s_{\beta^\lambda}, s_{\gamma^\lambda}, s_{\eta^\lambda}\right], \quad \lambda \geq 0$

(v) $\tilde{s}^{-1} = [s_\alpha, s_\beta, s_\gamma, s_\eta]^{-1} = \left[\frac{1}{s_\alpha}, \frac{1}{s_\beta}, \frac{1}{s_\gamma}, \frac{1}{s_\eta}\right] = \left[\frac{1}{s_{1/\eta}}, \frac{1}{s_{1/y}}, \frac{1}{s_{1/\beta}}, \frac{1}{s_{1/\alpha}}\right].$

Definition 4 [35]**:** Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1}, s_{\eta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2}, s_{\eta_2}]$ be two trapezoid fuzzy linguistic variables, then the degree of possibility, $p(\tilde{s}_1 \geq \tilde{s}_2)$, of $(\tilde{s}_1 \geq \tilde{s}_2)$ is defined as follows:

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \min\left\{\max\left\{\frac{(\gamma_1+\eta_1)-(\alpha_2+\beta_2)}{(\gamma_1+\eta_1)-(\alpha_1+\beta_1)+(\gamma_2+\eta_2)-(\alpha_2+\beta_2)}, 0\right\}, 1\right\}. \tag{3}$$

The characteristics of the possibility degree $p(\tilde{s}_1 \geq \tilde{s}_2)$ may be noted as follows [35]:

1. $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1$, $0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$.
2. $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$.

Especially, if $p(\tilde{s}_1 \geq \tilde{s}_2) = p(\tilde{s}_2 \geq \tilde{s}_1)$, then $p(\tilde{s}_1 \geq \tilde{s}_2) = p(\tilde{s}_2 \geq \tilde{s}_1) = \frac{1}{2}$.

Definition 5 Expected value of trapezoid fuzzy linguistic variable: Let $\tilde{s} = [s_\alpha, s_\beta, s_\gamma, s_\eta]$ be a trapezoid fuzzy linguistic variable, then the expected value of $\tilde{s}$ is defined as follows:

$$E(\tilde{s}) = \left(\frac{s_\alpha \oplus s_\beta \oplus s_\gamma \oplus s_\eta}{4}\right). \tag{4}$$

The prioritized weighted average (PWA) operator was originally introduced by Yager [42,43] as follows:

Definition 6. PWA operator [42,43]**:** Let $G = \{G_1, G_2, \ldots, G_n\}$ be a collection of attributes and let there be a prioritization between the attributes expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \ldots \succ G_n$, indicating that attribute G_i has a higher priority than G_j, if $i < j$. Also, let $G_i(x)$ be the performance value of any alternative x under attribute G_i and satisfies $G_i(x) \in [0, 1]$. If

$$\mathrm{PWA}(G_1(x), G_2(x), \ldots, G_n(x)) = \sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} G_i(x), \tag{5}$$

where $T_i = \prod_{j=1}^{i-1} G_j(x), \quad i = 2, 3, \ldots, n$, $T_1 = 1$, then PWA($G_1(x)$, $G_2(x)$,…,$G_n(x)$) is called the PWA operator.

In the next section, to aggregate the trapezoid fuzzy linguistic information, we propose some prioritized weighted aggregation operators such as the trapezoid fuzzy linguistic prioritized weighted average (TFLPWA) operator, the trapezoid fuzzy linguistic prioritized weighted geometric (TFLPWG) operator, and the trapezoid fuzzy linguistic prioritized weighted harmonic average (TFLPWHA) operator with properties.

Trapezoid fuzzy linguistic prioritized weighted aggregation operators

Trapezoid fuzzy linguistic prioritized weighted average operator

Based on Definition 6, we give definition of the TFLPWA operator as follows:

Definition 7. Trapezoid fuzzy linguistic prioritized weighted average operator: Given a set of trapezoid fuzzy linguistic variables, $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right], i = 1, 2, \ldots, n$, the TFLPWA operator is defined as follows:

$$\mathrm{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \overset{n}{\underset{i=1}{\oplus}} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \tilde{s} \right)_i = \frac{T_1}{\sum_{i=1}^{n} T_i} \tilde{s}_1 \oplus \frac{T_2}{\sum_{i=1}^{n} T_i} \tilde{s}_2 \oplus \cdots \oplus \frac{T_n}{\sum_{i=1}^{n} T_i} \tilde{s}_n, \tag{6}$$

where $T_i = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j\right)\right)}{t}, \quad i = 2, 3, \ldots, n,\ T_1 = 1$ and $E\left(\tilde{s}_j\right)$ is the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$, $I\left(E\left(\tilde{s}_j\right)\right)$ is the subscript of $E\left(\tilde{s}_j\right)$.

Note 1: If the priority levels of the aggregated arguments reduce to the same level, then the TFLPWA operator reduces to the trapezoid fuzzy linguistic weighted average (TFLWA) operator [34,35]:

$$\mathrm{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(w_1 \tilde{s}_1 \oplus w_2 \tilde{s}_2 \oplus \cdots \oplus w_n \tilde{s}_n\right). \tag{7}$$

Next, based on the operational laws of TFLVs, we can easily prove the following theorem:

Theorem 1. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right],\ i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, then the aggregated value by using the TFLPWA operator is also a trapezoid fuzzy linguistic variable, and

$$\begin{aligned} \mathrm{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \overset{n}{\underset{i=1}{\oplus}} \frac{T_i}{\sum_{i=1}^{n} T_i} \tilde{s}_i, \\ &= \left[\overset{n}{\underset{i=1}{\oplus}} \frac{T_i}{\sum_{i=1}^{n} T_i} s_{\alpha_i}, \overset{n}{\underset{i=1}{\oplus}} \frac{T_i}{\sum_{i=1}^{n} T_i} s_{\beta_i}, \overset{n}{\underset{i=1}{\oplus}} \frac{T_i}{\sum_{i=1}^{n} T_i} s_{\gamma_i}, \overset{n}{\underset{i=1}{\oplus}} \frac{T_i}{\sum_{i=1}^{n} T_i} s_{\eta_i} \right], \end{aligned} \tag{8}$$

where $T_i = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j\right)\right)}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E\left(\tilde{s}_j\right)$ is the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$ and $I\left(E\left(\tilde{s}_j\right)\right)$ is the subscript of $E\left(\tilde{s}_j\right)$.

Properties of TFLPWA operator

P1. (*Idempotency*): Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right],\ i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j\right)\right)}{t}, i = 2, 3, \ldots, n,\ T_1 = 1$, $E\left(\tilde{s}_j\right)$ be the expected value of

$\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$ and $I\left(E\left(\tilde{s}_j\right)\right)$ be the subscript of $E\left(\tilde{s}_j\right)$. If all the trapezoid fuzzy linguistic variables $\tilde{s}_i$, $i = 1, 2, \ldots, n$, are equal, i.e., $\tilde{s}_i = \tilde{s} = \left[s_\alpha, s_\beta, s_\gamma, s_\eta\right] \forall i$, then

$$\text{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}. \tag{9}$$

P2. (*Boundedness*): Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$, $i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j\right)\right)}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E\left(\tilde{s}_j\right)$ be the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$ and $I\left(E\left(\tilde{s}_j\right)\right)$ be the subscript of $E\left(\tilde{s}_j\right)$. Also, let

$$\tilde{s}^- = \min_i \tilde{s}_i = \left[\min_i s_{\alpha_i},\ \min_i s_{\beta_i},\ \min_i s_{\gamma_i},\ \min_i s_{\eta_i}\right], \tag{10}$$

and

$$\tilde{s}^+ = \max_i \tilde{s}_i = \left[\max_i s_{\alpha_i},\ \max_i s_{\beta_i},\ \max_i s_{\gamma_i},\ \max_i s_{\eta_i}\right]. \tag{11}$$

Then,

$$\tilde{s}^- \leq \text{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \tilde{s}^+. \tag{12}$$

P3. (*Monotonicity*): Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$ and $\tilde{s}_i' = \left[s_{\alpha_i}', s_{\beta_i}', s_{\gamma_i}', s_{\eta_i}'\right], i = 2, 3, \ldots, n$, be two sets of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j\right)\right)}{t}$, $\quad T_i' = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j'\right)\right)}{t}$, $i = 2, 3, \ldots, n$, $T_1 = T_1' = 1$, $E\left(\tilde{s}_j\right)$ be the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$, $E\left(\tilde{s}_j'\right)$ be the expected value of $\tilde{s}_j' = \left[s_{\alpha_j}', s_{\beta_j}', s_{\gamma_j}', s_{\eta_j}'\right]$, $I\left(E\left(\tilde{s}_j\right)\right)$ be the subscript of $E\left(\tilde{s}_j\right)$, $I\left(E\left(\tilde{s}_j'\right)\right)$ be the subscript of $E\left(\tilde{s}_j'\right)$. If $\tilde{s}_i \leq \tilde{s}_i'$ for all i, then

$$\text{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \text{TFLPWA}\left(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n'\right). \tag{13}$$

Trapezoid fuzzy linguistic prioritized weighted geometric operator

Based on TFLPWA operator and the geometric mean, here, we give the definition of the TFLPWG operator as follows:

Definition 8. Trapezoid fuzzy linguistic prioritized weighted geometric operator: Given a set of trapezoid fuzzy linguistic variables, $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$, $\quad i = 1, 2, \ldots, n$, the TFLPWG operator is defined as follows:

$$\begin{aligned} \text{TFLPWG}(\widetilde{s}_1, \widetilde{s}_2, \ldots, \widetilde{s}_n) &= \bigotimes_{i=1}^{n} (\widetilde{s}_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} \\ &= \left((\widetilde{s}_1)^{\frac{T_1}{\sum_{i=1}^{n} T_i}} \otimes (\widetilde{s}_2)^{\frac{T_2}{\sum_{i=1}^{n} T_i}} \otimes \cdots \otimes (\widetilde{s}_n)^{\frac{T_n}{\sum_{i=1}^{n} T_i}} \right), \end{aligned} \tag{14}$$

where $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, \quad i = 2, 3, \ldots, n$, $T_1 = 1$, $E(\tilde{s}_j)$ is the expected value of $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}]$ and $I(E(\tilde{s}_j))$ is the subscript of $E(\tilde{s}_j)$.

Note 2: If the priority levels of the aggregated arguments reduce to the same level, then the TFLPWG operator reduces to the trapezoid fuzzy linguistic weighted geometric (TFLWG) operator [36]:

$$\mathrm{TFLPWG}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = (\tilde{s}_1)^{w_1} \otimes (\tilde{s}_2)^{w_2} \otimes \cdots \otimes (\tilde{s}_n)^{w_n}. \tag{15}$$

Next, based on the operational laws of TFLVs, we can prove a result in the following theorem:

Theorem 2. Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}]$, $i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, then the aggregated value by using the TFLPWG operator is also a trapezoid fuzzy linguistic variable, and

$$\begin{aligned}\mathrm{TFLPWG}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \bigotimes_{i=1}^{n} (\tilde{s}_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}, \\ &= \left[\bigotimes_{i=1}^{n} (s_{\alpha_i})^{\frac{T_i}{\sum_{i=1}^{n} T_i}}, \bigotimes_{i=1}^{n} (s_{\beta_i})^{\frac{T_i}{\sum_{i=1}^{n} T_i}}, \bigotimes_{i=1}^{n} s_{\gamma_i}{}^{\frac{T_i}{\sum_{i=1}^{n} T_i}}, \bigotimes_{i=1}^{n} (s_{\eta_i})^{\frac{T_i}{\sum_{i=1}^{n} T_i}} \right],\end{aligned} \tag{16}$$

where $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E(\tilde{s}_j)$ is the expected value of $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}]$ and $I(E(\tilde{s}_j))$ is the subscript of $E(\tilde{s}_j)$.

Properties of TFLPWG operator

P1. (*Idempotency*): Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}]$, $\quad i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}]$ and $I(E(\tilde{s}_j))$ be the subscript of $E(\tilde{s}_j)$. If all the trapezoid fuzzy linguistic variables $\tilde{s}_i$, i = 1, 2,..., n, are equal, i.e., $\tilde{s}_i = \tilde{s} = [s_\alpha, s_\beta, s_\gamma, s_\eta] \forall i$, then

$$\mathrm{TFLPWG}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}. \tag{17}$$

P2. (*Boundedness*): Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}]$, $i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}]$ and $I(E(\tilde{s}_j))$ be the subscript of $E(\tilde{s}_j)$. Also, let

$$\tilde{s}^- = \min_i \tilde{s}_i = \left[\min_i s_{\alpha_i}, \min_i s_{\beta_i}, \min_i s_{\gamma_i}, \min_i s_{\eta_i} \right], \tag{18}$$

and

$$\tilde{s}^{+} = \max_i \tilde{s}_i = \left[\max_i s_{\alpha_i},\ \max_i s_{\beta_i},\ \max_i s_{\gamma_i},\ \max_i s_{\eta_i}\right]. \tag{19}$$

Then

$$\tilde{s}^{-} \leq \mathrm{TFLPWG}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \tilde{s}^{+}. \tag{20}$$

P3. (*Monotonicity*): Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$ and $\tilde{s}_i^{'} = \left[s_{\alpha_i}^{'}, s_{\beta_i}^{'}, s_{\gamma_i}^{'}, s_{\eta_i}^{'}\right], i = 2, 3, \ldots, n$, be two sets of trapezoid fuzzy linguistic variables, $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, T_i^{'} = \prod_{j=1}^{i-1} \frac{I\left(E\left(\tilde{s}_j^{'}\right)\right)}{t}$, $i = 2, 3, \ldots, n$, $T_1 = T_1^{'} = 1$, $E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$, $E\left(\tilde{s}_j^{'}\right)$ be the expected value of $\tilde{s}_j^{'} = \left[s_{\alpha_j}^{'}, s_{\beta_j}^{'}, s_{\gamma_j}^{'}, s_{\eta_j}^{'}\right]$, $I(E(\tilde{s}_j))$ the subscript of $E(\tilde{s}_j)$, $I\left(E\left(\tilde{s}_j^{'}\right)\right)$ be the subscript of $E\left(\tilde{s}_j^{'}\right)$. If $\tilde{s}_i \leq \tilde{s}_i^{'}$ for all i, then

$$\mathrm{TFLPWG}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \mathrm{TFLPWG}\left(\tilde{s}_1^{'}, \tilde{s}_2^{'}, \ldots, \tilde{s}_n^{'}\right). \tag{21}$$

Trapezoid fuzzy linguistic prioritized weighted harmonic average operator

Based on TFLPWA operator and the harmonic average, here, we give the definition of the TFLPWHA operator as follows:

Definition 9. Trapezoid fuzzy linguistic prioritized weighted harmonic average operator: Given a set of trapezoid fuzzy linguistic variables, $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$, $i = 1, 2, \ldots, n$, the TFLPWHA operator is defined as follows:

$$\mathrm{TFLPWA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \left(\bigoplus_{i=1}^{n} \frac{\frac{T_i}{\sum_{i=1}^{n} T_i}}{\tilde{s}_i}\right)^{-1} = \frac{1}{\frac{\frac{T_1}{\sum_{i=1}^{n} T_i}}{\tilde{s}_1} \oplus \frac{\frac{T_2}{\sum_{i=1}^{n} T_i}}{\tilde{s}_2} \oplus \cdots \oplus \frac{\frac{T_n}{\sum_{i=1}^{n} T_i}}{\tilde{s}_n}}, \tag{22}$$

where $T_i = \prod_{j=1}^{i-1} \frac{I(E(\tilde{s}_j))}{t}, i = 2, 3, \ldots, n$, $T_1 = 1$, $E(\tilde{s}_j)$ is the expected value of $\tilde{s}_j = \left[s_{\alpha_j}, s_{\beta_j}, s_{\gamma_j}, s_{\eta_j}\right]$ and $I(E(\tilde{s}_j))$ is the subscript of $E(\tilde{s}_j)$.

Note 3: If the priority levels of the aggregated arguments reduce to the same level, then the TFLPWHA operator reduces to the TFLWHA operator [38]:

$$\mathrm{TFLPWHA}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_1} \oplus \frac{w_2}{\tilde{s}_2} \oplus \cdots \frac{w_n}{\tilde{s}_n}}. \tag{23}$$

Next, based on the operational laws of TFLVs, we can prove a result in the following theorem:

Theorem 3. Let $\tilde{s}_i = \left[s_{\alpha_i}, s_{\beta_i}, s_{\gamma_i}, s_{\eta_i}\right]$, $i = 1, 2, \ldots, n$, be a set of trapezoid fuzzy linguistic variables, then the aggregated value by using the TFLPWHA operator is also a trapezoid fuzzy linguistic variable, and

$$\mathrm{TFLPWA}(\tilde{s}_1,\tilde{s}_2,\dots,\tilde{s}_n)=\left(\mathop{\oplus}_{i=1}^{n}\frac{\frac{T_i}{\sum_{i=1}^{n}T_i}}{\tilde{s}_i}\right)^{-1},$$

$$=\left[\left(\mathop{\oplus}_{i=1}^{n}\frac{\frac{T_i}{\sum_{i=1}^{n}T_i}}{s_{\alpha_i}}\right)^{-1},\left(\mathop{\oplus}_{i=1}^{n}\frac{\frac{T_i}{\sum_{i=1}^{n}T_i}}{s_{\beta_i}}\right)^{-1},\left(\mathop{\oplus}_{i=1}^{n}\frac{\frac{T_i}{\sum_{i=1}^{n}T_i}}{s_{\gamma_i}}\right)^{-1},\left(\mathop{\oplus}_{i=1}^{n}\frac{\frac{T_i}{\sum_{i=1}^{n}T_i}}{s_{\eta_i}}\right)^{-1}\right], \tag{24}$$

where $T_i=\prod_{j=1}^{i-1}\frac{I\left(E(\tilde{s}_j)\right)}{t}, i=2,3,\dots,n,\ T_1=1$ and $E(\tilde{s}_j)$ is the expected value of $\tilde{s}_j=\left[s_{\alpha_j},s_{\beta_j},s_{\gamma_j},s_{\eta_j}\right]$, $I\left(E(\tilde{s}_j)\right)$ is the subscript of $E(\tilde{s}_j)$.

Properties of TFLPWHA operator

P1. (*Idempotency*): Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i},s_{\gamma_i},s_{\eta_i}\right],\ i=1,2,\dots,n$, be a set of trapezoid fuzzy linguistic variables, $T_i=\prod_{j=1}^{i-1}\frac{I\left(E(\tilde{s}_j)\right)}{t}, i=2,3,\dots,n,\ T_1=1,\ E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j=\left[s_{\alpha_j},s_{\beta_j},s_{\gamma_j},s_{\eta_j}\right]$ and $I\left(E(\tilde{s}_j)\right)$ be the subscript of $E(\tilde{s}_j)$. If all the trapezoid fuzzy linguistic variables $\tilde{s}_i$, $i = 1, 2,\dots, n$, are equal, i.e., $\tilde{s}_i=\tilde{s}=\left[s_{\alpha},s_{\beta},s_{\gamma},s_{\eta}\right]\ \forall i$, then

$$\mathrm{TFLPWHA}(\tilde{s}_1,\tilde{s}_2,\dots,\tilde{s}_n)=\tilde{s}. \tag{25}$$

P2. (*Boundedness*): Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i},s_{\gamma_i},s_{\eta_i}\right],\ i=1,2,\dots,n$, be a set of trapezoid fuzzy linguistic variables, $T_i=\prod_{j=1}^{i-1}\frac{I\left(E(\tilde{s}_j)\right)}{t}, i=2,3,\dots,n,\ T_1=1,\ E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j=\left[s_{\alpha_j},s_{\beta_j},s_{\gamma_j},s_{\eta_j}\right]$ and $I\left(E(\tilde{s}_j)\right)$ be the subscript of $E(\tilde{s}_j)$. Also, let

$$\tilde{s}^-=\min_i\tilde{s}_i=\left[\min_i s_{\alpha_i},\ \min_i s_{\beta_i},\ \min_i s_{\gamma_i},\ \min_i s_{\eta_i}\right], \tag{26}$$

and

$$\tilde{s}^+=\max_i\tilde{s}_i=\left[\max_i s_{\alpha_i},\ \max_i s_{\beta_i},\ \max_i s_{\gamma_i},\ \max_i s_{\eta_i}\right]. \tag{27}$$

Then

$$\tilde{s}^-\le\mathrm{TFLPWHA}(\tilde{s}_1,\tilde{s}_2,\dots,\tilde{s}_n)\le\tilde{s}^+. \tag{28}$$

P3. (*Monotonicity*): Let $\tilde{s}_i=\left[s_{\alpha_i},s_{\beta_i},s_{\gamma_i},s_{\eta_i}\right]$ and $\tilde{s}_i'=\left[s_{\alpha_i}',s_{\beta_i}',s_{\gamma_i}',s_{\eta_i}'\right],\ i=1,2,\dots,n$, be two sets of trapezoid fuzzy linguistic variables, $T_i=\prod_{j=1}^{i-1}\frac{I\left(E(\tilde{s}_j)\right)}{t}, T_i'=\prod_{j=1}^{i-1}\frac{I\left(E\left(\tilde{s}_j'\right)\right)}{t}$, $i=2,3,\dots,n,\ T_1=T_1'=1$, $E(\tilde{s}_j)$ be the expected value of $\tilde{s}_j=\left[s_{\alpha_j},s_{\beta_j},s_{\gamma_j},s_{\eta_j}\right]$, $E\left(\tilde{s}_j'\right)$ be the expected value of $\tilde{s}_j^{'}=\left[s_{\alpha_j}^{'},s_{\beta_j}^{'},s_{\gamma_j}^{'},s_{\eta_j}^{'}\right]$, $I(E(\tilde{s}_j))$ the subscript of $E(\tilde{s}_j)$, $I\left(E\left(\tilde{s}_j^{'}\right)\right)$ be the subscript of $E\left(\tilde{s}_j^{'}\right)$. If $\tilde{s}_i\leq\tilde{s}_i^{'}$ for all i, then

$$\mathrm{TFLPWHA}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\leq\mathrm{TFLPWHA}\left(\tilde{s}_1^{'},\tilde{s}_2^{'},\ldots,\tilde{s}_n^{'}\right). \tag{29}$$

In the following section, we suggest the application of the proposed operators to solve multiple-attribute decision-making problems with trapezoid fuzzy linguistic information.

An approach to multiple attribute group decision-making with trapezoid fuzzy uncertain linguistic information

Let us consider a multiple attribute group decision-making problem involving a set of alternatives $X=\{X_1, X_2,\ldots, X_m\}$ to be considered under a set of attributes $G=\{G_1, G_2,\ldots, G_n\}$ and let there be a prioritization between the attributes expressed by the linear ordering $G_1\succ G_2\succ\cdots\succ G_n$ (indicating that attribute G_j has a higher priority than G_l, if $j<l$), and let $D=\{D_1, D_2,\ldots, D_q\}$ be the set of decision makers and let there be a prioritization between the decision makers expressed by the linear ordering $D_1\succ D_2\succ\cdots\succ D_q$, indicating decision maker D_η has a higher priority than D_ς, if $\eta<\varsigma$. Let $\tilde{R}^{(k)}=\left(\tilde{r}_{ij}{}^{(k)}\right)_{m\times n}=\left(\left[s_{\alpha_{ij}}^{(k)},s_{\beta_{ij}}^{(k)},s_{\gamma_{ij}}^{(k)},s_{\eta_{ij}}^{(k)}\right]\right)_{m\times n}$ be a trapezoid fuzzy linguistic decision matrix, where $\tilde{r}_{ij}{}^{(k)}\in\tilde{S}$ is an attribute value, which takes the form of trapezoid fuzzy linguistic variables, provided by the decision maker $D_k\in D$, for the alternative $X_i\in X$ with respect to the attribute $G_j\in G$.

Using the TFLPWA (or TFLPWG or TFLPWHA) operator, we now formulate an algorithm to solve multiple attribute group decision-making problems with trapezoid fuzzy linguistic information:

Step 1. Calculate the values of $T_{ij}^{(k)}$ $(k=1,2,\ldots,q)$ as follows:

$$T_{ij}^{(k)}=\prod_{\gamma=1}^{k-1}\frac{I\left(E\left(\tilde{r}_{ij}^{\gamma}\right)\right)}{t}\quad(k=2,3,\ldots,q), \tag{30}$$

$$T_{ij}^{(1)}=1. \tag{31}$$

Step 2. Utilize appropriately the TFLPWA operator:

$$\begin{aligned}\tilde{r}_{ij}&=\left(\tilde{s}_{\alpha_{ij}},\tilde{s}_{\beta_{ij}},\tilde{s}_{\gamma_{ij}},\tilde{s}_{\eta_{ij}}\right)\\&=\mathrm{TFLPWA}\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\ldots,\tilde{r}_{ij}^{(q)}\right)\\&=\frac{T_{ij}^{(1)}}{\sum_{k=1}^{q}T_{ij}^{(k)}}\tilde{r}_{ij}^{(1)}\oplus\frac{T_{ij}^{(2)}}{\sum_{k=1}^{q}T_{ij}^{(k)}}\tilde{r}_{ij}^{(2)}\oplus\cdots\oplus\frac{T_{ij}^{(q)}}{\sum_{k=1}^{q}T_{ij}^{(k)}}\tilde{r}_{ij}^{(q)}\\&=\left[\bigoplus_{k=1}^{q}\frac{T_{ij}^{(k)}s_{\alpha_{ij}}^{(k)}}{\sum_{k=1}^{q}T_{ij}^{(k)}},\bigoplus_{k=1}^{q}\frac{T_{ij}^{(k)}s_{\beta_{ij}}^{(k)}}{\sum_{k=1}^{q}T_{ij}^{(k)}},\bigoplus_{k=1}^{q}\frac{T_{ij}^{(k)}s_{\gamma_{ij}}^{(k)}}{\sum_{k=1}^{q}T_{ij}^{(k)}},\bigoplus_{k=1}^{q}\frac{T_{ij}^{(k)}s_{\eta_{ij}}^{(k)}}{\sum_{k=1}^{q}T_{ij}^{(k)}}\right],\end{aligned} \tag{32}$$

or the TFLPWG operator

$$\tilde{r}_{ij} = \left(\tilde{s}_{\alpha_{ij}}, \tilde{s}_{\beta_{ij}}, \tilde{s}_{\gamma_{ij}}, \tilde{s}_{\eta_{ij}}\right) = \mathrm{TFLPWG}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(q)}\right)$$
$$= \left(\tilde{r}_{ij}^{(1)}\right)^{\frac{T_{ij}^{(1)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} \otimes \left(\tilde{r}_{ij}^{(2)}\right)^{\frac{T_{ij}^{(2)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}} \otimes \cdots \otimes \left(\tilde{r}_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}$$
$$= \left[\bigotimes_{k=1}^{q} \left(s_{\alpha_{ij}}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}, \bigotimes_{k=1}^{q} \left(s_{\beta_{ij}}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}, \bigotimes_{k=1}^{q} \left(s_{\gamma_{ij}}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}, \bigotimes_{k=1}^{q} \left(s_{\eta_{ij}}^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}\right], \tag{33}$$

or the TFLPWHA operator

$$\tilde{r}_{ij} = \left(\tilde{s}_{\alpha_{ij}}, \tilde{s}_{\beta_{ij}}, \tilde{s}_{\gamma_{ij}}, \tilde{s}_{\eta_{ij}}\right) = \mathrm{TFLPWHA}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(q)}\right)$$
$$= \left(\frac{\frac{T_{ij}^{(1)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{\tilde{r}_{ij}^{(1)}}\right)^{-1} \oplus \left(\frac{\frac{T_{ij}^{(2)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{\tilde{r}_{ij}^{(2)}}\right)^{-1} \oplus \cdots \oplus \left(\frac{\frac{T_{ij}^{(q)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{\tilde{r}_{ij}^{(q)}}\right)^{-1}$$
$$= \left[\left(\bigoplus_{k=1}^{q} \frac{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{s_{\alpha_{ij}}^{(k)}}\right)^{-1}, \left(\bigoplus_{k=1}^{q} \frac{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{s_{\beta_{ij}}^{(k)}}\right)^{-1}, \left(\bigoplus_{k=1}^{q} \frac{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{s_{\gamma_{ij}}^{(k)}}\right)^{-1}, \left(\bigoplus_{k=1}^{q} \frac{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{q} T_{ij}^{(k)}}}{s_{\eta_{ij}}^{(k)}}\right)^{-1}\right], \tag{34}$$

to aggregate all the individual trapezoid fuzzy linguistic decision matrices $\tilde{R}^{(k)} = \left(\tilde{r}_{ij}{}^{(k)}\right)_{m\times n} = \left(\left[s_{\alpha_{ij}}^{(k)}, s_{\beta_{ij}}^{(k)}, s_{\gamma_{ij}}^{(k)}, s_{\eta_{ij}}^{(k)}\right]\right)_{m\times n}$ (k = 1, 2,…, q) into the collective trapezoid fuzzy linguistic decision matrix $\tilde{R}^* = \left(\tilde{r}_{ij}\right)_{m\times n} = \left(\left[s_{\alpha_{ij}}, s_{\beta_{ij}}, s_{\gamma_{ij}}, s_{\eta_{ij}}\right]\right)_{m\times n}$, i = 1, 2,…, m; j = 1, 2,…, n.

Step 3. Calculate the values T_{ij}, i = 1, 2,…, m, j = 1, 2,…, n, as follows:

$$T_{ij} = \prod_{v=1}^{j-1} \frac{I(E(\tilde{r}_{1v}))}{t}, \quad i = 1, 2, \ldots, m; \quad j = 2, 3, \ldots, n, \tag{35}$$

$$T_{i1} = 1, \quad i = 1, 2, \ldots, m. \tag{36}$$

Step 4. Aggregate all trapezoid fuzzy linguistic variables $\tilde{r}_{ij}, j = 1, 2, \ldots, n$, for each option X_i, i = 1, 2,…, m, by the TFLPWA operator:

$$\tilde{r}_i = \left(\tilde{s}_{\alpha_i}, \tilde{s}_{\beta_i}, \tilde{s}_{\gamma_i}, \tilde{s}_{\eta_i}\right) = \mathrm{TFLPWA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in})$$
$$= \frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{i1} \oplus \frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{i2} \oplus \cdots \oplus \frac{T_{in}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{in}$$
$$= \left[\bigoplus_{j=1}^{n} \frac{T_{ij}\left(s_{\alpha_{ij}}\right)}{\sum_{j=1}^{n} T_{ij}}, \bigoplus_{j=1}^{n} \frac{T_{ij}\left(s_{\beta_{ij}}\right)}{\sum_{j=1}^{n} T_{ij}}, \bigoplus_{j=1}^{n} \frac{T_{ij}\left(s_{\gamma_{ij}}\right)}{\sum_{j=1}^{n} T_{ij}}, \bigoplus_{j=1}^{n} \frac{T_{ij}\left(s_{\eta_{ij}}\right)}{\sum_{j=1}^{n} T_{ij}}\right], \quad i = 1, 2, \ldots, m \tag{37}$$

or the TFLPWG operator:

$$\begin{aligned}\tilde{r}_i &= \left(\tilde{s}_{\alpha_i}, \tilde{s}_{\beta_i}, \tilde{s}_{\gamma_i}, \tilde{s}_{\eta_i}\right) = \text{TFLPWG}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) \\ &= (\tilde{r}_{i1})^{\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}}} \otimes (\tilde{r}_{i2})^{\frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}}} \otimes \cdots \otimes (\tilde{r}_{in})^{\frac{T_{in}}{\sum_{j=1}^{n} T_{ij}}} \\ &= \left[\underset{j=1}{\overset{n}{\otimes}} \left(s_{\alpha_{ij}}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}, \underset{j=1}{\overset{n}{\otimes}} \left(s_{\beta_{ij}}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}, \underset{j=1}{\overset{n}{\otimes}} \left(s_{\gamma_{ij}}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}, \underset{j=1}{\overset{n}{\otimes}} \left(s_{\eta_{ij}}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} \right], \quad i = 1, 2, \ldots, m\end{aligned} \tag{38}$$

or the TFLPWHA operator:

$$\begin{aligned}\tilde{r}_i &= \left(\tilde{s}_{\alpha_i}, \tilde{s}_{\beta_i}, \tilde{s}_{\gamma_i}, \tilde{s}_{\eta_i}\right) = \text{TFLPWHA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) \\ &= \left(\frac{\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{i1}} \right)^{-1} \oplus \left(\frac{\frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{i2}} \right)^{-1} \oplus \cdots \oplus \left(\frac{\frac{T_{in}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{in}} \right)^{-1} \\ &= \left[\left(\underset{j=1}{\overset{n}{\oplus}} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{\left(s_{\alpha_{ij}}\right)} \right)^{-1}, \left(\underset{j=1}{\overset{n}{\oplus}} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{\left(s_{\beta_{ij}}\right)} \right)^{-1}, \left(\underset{j=1}{\overset{n}{\oplus}} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{\left(s_{\gamma_{ij}}\right)} \right)^{-1}, \left(\underset{j=1}{\overset{n}{\oplus}} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{\left(s_{\eta_{ij}}\right)} \right)^{-1} \right], \quad i = 1, 2, \ldots, m\end{aligned} \tag{39}$$

to derive the overall trapezoid fuzzy linguistic variables $\tilde{r}_i$, $i = 1, 2,\ldots, m$, of the options X_i, $i = 1, 2,\ldots, m$.

Step 5. Compare each $\tilde{r}_i$ with all $\tilde{r}_k$, $i, k = 1, 2,\ldots, m$, by (3). For simplicity, we let $p_{ik} = p$ $(r_i \geq r_k)$, and then construct the possibility matrix $P = (p_{ik})_{m \times m}$, where $p_{ik} \geq 0$, $p_{ik} + p_{ki} = 1$, $p_{ii} = 0.5$ $\forall i, k = 1, 2,\ldots, m$. Summing all the elements in each row of matrix P, get

$$p_i = \sum_{k=1}^{m} p_{ik}, \quad i = 1, 2, \ldots, m \tag{40}$$

Then, arrange the collective overall preference values $\tilde{r}_i$, $i = 1, 2,\ldots, m$, in descending order in accordance with the values of p_i, $i = 1, 2,\ldots, m$.

Step 6. Rank all the options X_i, $i = 1, 2,\ldots, m$, by the ranking of r_i, $i = 1, 2,\ldots, m$, and select the best one(s).

Step 7. End.

Numerical example

In order to demonstrate the applicability of the proposed method to multiple attribute group decision-making, we consider below a university faculty recruitment group decision-making problem.

Example: The department of mathematics in a university wants to appoint outstanding mathematics teachers. The appointment is done by a committee of three decision makers, President D_1, Dean of Academics D_2, and Human Resource Officer D_3. After preliminary screening, five teachers X_i, $i = 1, 2, 3, 4, 5$, remain for further evaluation. Panel of decision makers made strict evaluation for five teachers X_i, $i = 1, 2, 3, 4, 5$, according to the following four attributes: (1) G_1, the past experience, (2) G_2, the research capability, (3) G_3, subject knowledge, (4) G_4, the teaching skill. During this process, the university President has the absolute priority for decision-making, Dean of Academics comes next. The prioritization relationship for the attributes is as follows: $G_1 \succ G_2 \succ G_3 \succ G_4$. The three decision makers evaluated the teachers X_i, $i = 1, 2, 3, 4, 5$, with respect to the attributes G_j, $j = 1, 2, 3, 4$ by using the following linguistic scale:

$$S = \{ s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{slightly poor}, s_5 = \text{fair} \\ s_6 = \text{slightly good}, s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\},$$

and provided their evaluation values in terms of trapezoid fuzzy linguistic variables and constructed the following three trapezoid fuzzy linguistic decision matrices $\tilde{R}^{(q)} = \left(\tilde{r}_{ij}{}^{(q)}\right)_{5\times4}$, $q = 1, 2, 3$ (see Tables 1, 2, and 3).

Step 1: Utilize the expressions in (30) and (31) to calculate the $T_{ij}^{(1)}$, $T_{ij}^{(2)}$ and $T_{ij}^{(3)}$,

$$\left[T_{ij}^{(1)}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \left[T_{ij}^{(2)}\right] = \begin{bmatrix} 0.444 & 0.722 & 0.750 & 0.528 \\ 0.583 & 0.722 & 0.722 & 0.667 \\ 0.750 & 0.611 & 0.833 & 0.500 \\ 0.750 & 0.778 & 0.583 & 0.833 \\ 0.500 & 0.361 & 0.472 & 0.694 \end{bmatrix}$$

$$\left[T_{ij}^{(3)}\right] = \begin{bmatrix} 0.259 & 0.281 & 0.375 & 0.323 \\ 0.389 & 0.201 & 0.441 & 0.333 \\ 0.313 & 0.306 & 0.301 & 0.306 \\ 0.625 & 0.389 & 0.227 & 0.370 \\ 0.222 & 0.151 & 0.341 & 0.270 \end{bmatrix}.$$

Step 2: Utilize the TFLPWA operator (Equation 32) to aggregate all the individual decision matrices $R^{(q)}$, $q = 1, 2, 3$, into the collective decision matrix $R^* = \left(\tilde{r}_{ij}\right)_{5\times4} = \left(\left[s_{\alpha_{ij}}, s_{\beta_{ij}}, s_{\gamma_{ij}}, s_{\eta_{ij}}\right]\right)_{5\times4}$ and we get the following table (Table 4):

Step 3: Using expressions in (35) and (36) to calculate the T_{ij}, $i = 1, 2,\ldots, 5$; $j = 1, 2,\ldots, 4$, we get

$$\left[T_{ij}\right] = \begin{bmatrix} 1 & 0.5058 & 0.2732 & 0.1633 \\ 1 & 0.5697 & 0.3048 & 0.2071 \\ 1 & 0.6417 & 0.3693 & 0.2192 \\ 1 & 0.6886 & 0.4254 & 0.2231 \\ 1 & 0.4803 & 0.1958 & 0.1062 \end{bmatrix}.$$

Step 4: Aggregating all trapezoid fuzzy linguistic variables $\tilde{r}_{ij}, j = 1, 2, 3, 4$, by using the TFLPWA operator (Equation 37) to derive the overall trapezoid fuzzy linguistic variables $\tilde{r}_i, i = 1, 2, \ldots, 5$ of the teachers X_i, $i = 1, 2,\ldots, 5$:

Table 1 Trapezoid fuzzy linguistic decision matrix $\tilde{R}^{(1)}$

	G_1	G_2	G_3	G_4
X_1	$[s_2, s_3, s_5, s_6]$	$[s_4, s_5, s_8, s_9]$	$[s_5, s_6, s_7, s_9]$	$[s_3, s_4, s_5, s_7]$
X_2	$[s_3, s_5, s_6, s_7]$	$[s_5, s_6, s_7, s_8]$	$[s_4, s_5, s_8, s_9]$	$[s_4, s_5, s_7, s_8]$
X_3	$[s_4, s_6, s_8, s_9]$	$[s_4, s_5, s_6, s_7]$	$[s_6, s_7, s_8, s_9]$	$[s_3, s_4, s_5, s_6]$
X_4	$[s_5, s_6, s_7, s_9]$	$[s_4, s_7, s_8, s_9]$	$[s_3, s_5, s_6, s_7]$	$[s_6, s_7, s_8, s_9]$
X_5	$[s_3, s_4, s_5, s_6]$	$[s_1, s_2, s_4, s_6]$	$[s_2, s_4, s_5, s_6]$	$[s_4, s_5, s_7, s_9]$

Table 2 Trapezoid fuzzy linguistic decision matrix $\tilde{R}^{(2)}$

	G_1	G_2	G_3	G_4
X_1	$[s_3, s_5, s_6, s_7]$	$[s_2, s_3, s_4, s_5]$	$[s_3, s_4, s_5, s_6]$	$[s_4, s_5, s_6, s_7]$
X_2	$[s_4, s_5, s_7, s_8]$	$[s_1, s_2, s_3, s_4]$	$[s_4, s_5, s_6, s_7]$	$[s_3, s_4, s_5, s_6]$
X_3	$[s_2, s_3, s_4, s_6]$	$[s_2, s_4, s_5, s_7]$	$[s_1, s_3, s_4, s_5]$	$[s_4, s_5, s_6, s_7]$
X_4	$[s_6, s_7, s_8, s_9]$	$[s_3, s_4, s_5, s_6]$	$[s_2, s_3, s_4, s_5]$	$[s_2, s_3, s_5, s_6]$
X_5	$[s_1, s_3, s_5, s_7]$	$[s_2, s_3, s_4, s_6]$	$[s_5, s_6, s_7, s_8]$	$[s_2, s_3, s_4, s_5]$

$$\tilde{r}_1 = [s_{2.87}, s_{4.01}, s_{5.57}, s_{6.69}],$$
$$\tilde{r}_2 = [s_{3.29}, s_{4.58}, s_{5.92}, s_{6.95}],$$
$$\tilde{r}_3 = [s_{3.45}, s_{4.83}, s_{6.07}, s_{7.32}],$$
$$\tilde{r}_4 = [s_{3.66}, s_{5.15}, s_{6.35}, s_{7.53}],$$
$$\tilde{r}_5 = [s_{2.25}, s_{3.55}, s_{4.92}, s_{6.35}].$$

Step 5: Comparing each r_i with all r_k, i, $k = 1, 2, \ldots, 5$, by Equation (3) and let $p_{ik} = p(r_i \geq r_k)$, and then constructing the possibility matrix, we get

$$P = \begin{bmatrix} 0.5000 & 0.4229 & 0.3794 & 0.3301 & 0.5954 \\ 0.5771 & 0.5000 & 0.4540 & 0.4032 & 0.6753 \\ 0.6206 & 0.5460 & 0.5000 & 0.4499 & 0.7174 \\ 0.6699 & 0.5968 & 0.5501 & 0.5000 & 0.7666 \\ 0.4046 & 0.3247 & 0.2826 & 0.2334 & 0.5000 \end{bmatrix}.$$

Now, summing all the elements in each row of matrix P, we have

$$p_1 = 2.2278, p_2 = 2.6096, p_3 = 2.8339, p_4 = 3.0834, p_5 = 1.7453,$$

then

$$r_4 \succ r_3 \succ r_2 \succ r_1 \succ r_5.$$

Ranking all the teachers X_i, $i = 1, 2, 3, 4, 5$, in accordance with the values of r_i, $i = 1, 2, 3, 4, 5$, we have

$$X_4 \succ X_3 \succ X_2 \succ X_1 \succ X_5.$$

Thus X_4 is most desirable alternative.

Based on the TFLPWG operator, the decision steps are as follows:

Step 1': See step 1.

Step 2': Utilize the TFLPWG operator (Equation 33) to aggregate all the individual decision matrices $R^{(q)}$, $q = 1, 2, 3$, into the collective decision matrix $R^* = (\tilde{r}_{ij})_{5\times 4} = \left(\left[s_{\alpha_{ij}}, s_{\beta_{ij}}, s_{\gamma_{ij}}, s_{\eta_{ij}}\right]\right)_{5\times 4}$ and we get the following table (Table 5):

Table 3 Trapezoid fuzzy linguistic decision matrix $\tilde{R}^{(3)}$

	G_1	G_2	G_3	G_4
X_1	$[s_4, s_5, s_6, s_7]$	$[s_1, s_2, s_3, s_4]$	$[s_2, s_3, s_4, s_5]$	$[s_3, s_4, s_5, s_7]$
X_2	$[s_2, s_3, s_4, s_5]$	$[s_3, s_4, s_5, s_7]$	$[s_4, s_6, s_7, s_8]$	$[s_1, s_3, s_5, s_6]$
X_3	$[s_6, s_7, s_8, s_9]$	$[s_4, s_5, s_6, s_7]$	$[s_2, s_3, s_5, s_6]$	$[s_2, s_3, s_4, s_5]$
X_4	$[s_1, s_3, s_5, s_6]$	$[s_2, s_3, s_5, s_6]$	$[s_4, s_5, s_6, s_7]$	$[s_2, s_3, s_4, s_5]$
X_5	$[s_2, s_4, s_5, s_6]$	$[s_4, s_6, s_7, s_8]$	$[s_3, s_4, s_5, s_6]$	$[s_4, s_5, s_6, s_7]$

Table 4 Collective trapezoid fuzzy linguistic decision matrix R^* (for TFLPWA operator)

	G_1	G_2	G_3	G_4
X_1	$[s_{2.56}, s_{3.83}, s_{5.41}, s_{6.41}]$	$[s_{2.86}, s_{3.86}, s_{5.86}, s_{6.86}]$	$[s_{3.76}, s_{4.76}, s_{5.76}, s_{7.24}]$	$[s_{3.29}, s_{4.29}, s_{5.29}, s_{7.00}]$
X_2	$[s_{3.10}, s_{4.61}, s_{5.90}, s_{6.90}]$	$[s_{3.29}, s_{4.29}, s_{5.29}, s_{6.39}]$	$[s_{4.00}, s_{5.20}, s_{7.13}, s_{8.13}]$	$[s_{3.17}, s_{4.33}, s_{6.00}, s_{7.00}]$
X_3	$[s_{3.58}, s_{5.06}, s_{6.55}, s_{7.91}]$	$[s_{3.36}, s_{4.68}, s_{5.68}, s_{7.00}]$	$[s_{3.48}, s_{4.87}, s_{6.01}, s_{7.01}]$	$[s_{3.11}, s_{4.11}, s_{5.11}, s_{6.11}]$
X_4	$[s_{4.26}, s_{5.53}, s_{6.79}, s_{8.21}]$	$[s_{3.28}, s_{5.20}, s_{6.38}, s_{7.38}]$	$[s_{2.80}, s_{4.36}, s_{5.36}, s_{6.36}]$	$[s_{3.82}, s_{4.82}, s_{6.19}, s_{7.19}]$
X_5	$[s_{2.29}, s_{3.71}, s_{5.00}, s_{6.29}]$	$[s_{1.54}, s_{2.64}, s_{4.30}, s_{6.20}]$	$[s_{2.97}, s_{4.52}, s_{5.52}, s_{6.52}]$	$[s_{3.73}, s_{4.29}, s_{5.80}, s_{7.31}]$

Step 3': Utilize expressions in (35) and (36) to calculate the T_{ij}, $i = 1, 2,\ldots, 5$; $j = 1, 2,\ldots, 4$, we get

$$[T_{ij}] = \begin{bmatrix} 1 & 0.4986 & 0.2515 & 0.1454 \\ 1 & 0.5597 & 0.2705 & 0.1828 \\ 1 & 0.6131 & 0.3496 & 0.1892 \\ 1 & 0.6525 & 0.3906 & 0.2009 \\ 1 & 0.4728 & 0.1867 & 0.0992 \end{bmatrix}.$$

Step 4': Aggregating all trapezoid fuzzy linguistic variables $\tilde{r}_{ij}, j = 1, 2, 3, 4,$ by using the TFLPWG operator (Equation 38) to derive the overall trapezoid fuzzy linguistic variables $\tilde{r}_i, i = 1, 2, \ldots, 5$ of the teachers X_i, $i = 1, 2,\ldots, 5$:

$$\tilde{r}_1 = [s_{2.68}, s_{3.84}, s_{5.42}, s_{6.55}], \tilde{r}_2 = [s_{2.99}, s_{4.37}, s_{5.70}, s_{6.76}], \tilde{r}_3 = [s_{3.12}, s_{4.62}, s_{5.86}, s_{7.22}],$$
$$\tilde{r}_4 = [s_{3.23}, s_{4.89}, s_{6.21}, s_{7.40}]\tilde{r}_5 = [s_{1.97}, s_{3.39}, s_{4.86}, s_{6.31}].$$

Step 5': Comparing each r_i with all r_k, $i, k = 1, 2,\ldots,5$, by Equation (3) and let $p_{ik} = p(r_i \geq r_k)$, and then constructing the possibility matrix, we get

$$P = \begin{bmatrix} 0.5000 & 0.4370 & 0.3920 & 0.3519 & 0.5870 \\ 0.5630 & 0.5000 & 0.4521 & 0.4098 & 0.6508 \\ 0.6080 & 0.5479 & 0.5000 & 0.4580 & 0.6924 \\ 0.6481 & 0.5902 & 0.5420 & 0.5000 & 0.7301 \\ 0.4130 & 0.3492 & 0.3076 & 0.2699 & 0.5000 \end{bmatrix}.$$

Now, summing all the elements in each row of matrix P, we have

$$p_1 = 2.2679, p_2 = 2.5757, p_3 = 2.8063, p_4 = 3.0122, p_5 = 1.8397,$$

then

$$r_4 \succ r_3 \succ r_2 \succ r_1 \succ r_5.$$

Ranking all the teachers X_i, $i = 1, 2, 3, 4, 5$, in accordance with the values of r_i, $i = 1, 2, 3, 4, 5$, we have

$$X_4 \succ X_3 \succ X_2 \succ X_1 \succ X_5.$$

Hence, X_4 is most desirable alternative.

Table 5 Collective trapezoid fuzzy linguistic decision matrix R^* (for TFLPWG operator)

	G_1	G_2	G_3	G_4
X_1	$[s_{2.47}, s_{3.70}, s_{5.39}, s_{6.39}]$	$[s_{2.57}, s_{3.66}, s_{5.43}, s_{6.50}]$	$[s_{3.55}, s_{4.60}, s_{5.63}, s_{7.03}]$	$[s_{3.26}, s_{4.26}, s_{5.27}, s_{7.00}]$
X_2	$[s_{3.02}, s_{4.52}, s_{5.80}, s_{6.81}]$	$[s_{2.59}, s_{3.81}, s_{4.92}, s_{6.08}]$	$[s_{4.00}, s_{5.18}, s_{7.07}, s_{8.08}]$	$[s_{2.89}, s_{4.26}, s_{5.92}, s_{6.93}]$
X_3	$[s_{3.31}, s_{4.77}, s_{6.22}, s_{7.77}]$	$[s_{3.21}, s_{4.66}, s_{5.66}, s_{7.00}]$	$[s_{2.55}, s_{4.46}, s_{5.71}, s_{6.76}]$	$[s_{3.03}, s_{4.05}, s_{5.06}, s_{6.07}]$
X_4	$[s_{3.47}, s_{5.25}, s_{6.68}, s_{8.09}]$	$[s_{3.19}, s_{4.92}, s_{6.21}, s_{7.23}]$	$[s_{2.73}, s_{4.24}, s_{5.27}, s_{6.28}]$	$[s_{3.29}, s_{4.41}, s_{5.96}, s_{7.00}]$
X_5	$[s_{2.07}, s_{3.68}, s_{5.00}, s_{6.27}]$	$[s_{1.36}, s_{2.46}, s_{4.23}, s_{6.17}]$	$[s_{2.74}, s_{4.45}, s_{5.46}, s_{6.47}]$	$[s_{3.64}, s_{4.17}, s_{5.62}, s_{7.06}]$

Based on the TFLPWHA operator, the decision steps are as follows:

Step 1": See step 1.

Step 2": Utilize the TFLPWHA operator (Equation 34) to aggregate all the individual decision matrices $R^{(q)}$, $q = 1,2,3$, into the collective decision matrix $R^* = \left(\tilde{r}_{ij}\right)_{5\times4} = \left(\left[s_{\alpha_{ij}}, s_{\beta_{ij}}, s_{\gamma_{ij}}, s_{\eta_{ij}}\right]\right)_{5\times4}$ and we get the following table (Table 6):

Step 3": Utilize expressions in (35) and (36) to calculate the T_{ij}, $i = 1, 2,\ldots,5$; $j = 1, 2,\ldots,4$, it gives

$$[T_{ij}] = \begin{bmatrix} 1 & 0.4925 & 0.2308 & 0.1289 \\ 1 & 0.5486 & 0.2371 & 0.1603 \\ 1 & 0.5836 & 0.3291 & 0.1635 \\ 1 & 0.6097 & 0.3533 & 0.1781 \\ 1 & 0.4647 & 0.1796 & 0.0936 \end{bmatrix}.$$

Step 4": Aggregating all trapezoid fuzzy linguistic variables $\tilde{r}_{ij}, j = 1, 2, \ldots, n$ by using the TFLPWHA operator (Equation 39) to derive the overall trapezoid fuzzy linguistic variables $\tilde{r}_i, i = 1, 2, \ldots, m$ of the teachers X_i:

$$\tilde{r}_1 = [s_{2.48}, s_{3.68}, s_{5.28}, s_{6.41}], \tilde{r}_2 = [s_{2.68}, s_{4.11}, s_{5.45}, s_{6.56}], \tilde{r}_3 = [s_{2.77}, s_{4.42}, s_{5.66}, s_{7.11}]$$
$$\tilde{r}_4 = [s_{2.75}, s_{4.69}, s_{6.12}, s_{7.29}], \tilde{r}_5 = [s_{1.70}, s_{3.25}, s_{4.82}, s_{6.29}].$$

Step 5": Comparing each r_i with all r_k, $i, k = 1, 2,\ldots,5$, by Equation (3) and let $p_{ik} = p(r_i \geq r_k)$, and then constructing the possibility matrix

$$P = \begin{bmatrix} 0.5000 & 0.4558 & 0.4050 & 0.3696 & 0.5766 \\ 0.5442 & 0.5000 & 0.4463 & 0.4084 & 0.6204 \\ 0.5950 & 0.5537 & 0.5000 & 0.4615 & 0.6661 \\ 0.6304 & 0.5916 & 0.5385 & 0.5000 & 0.6974 \\ 0.4234 & 0.3796 & 0.3339 & 0.3026 & 0.5000 \end{bmatrix}.$$

Now, summing all the elements in each row of matrix P, we have

$$p_1 = 2.3070, p_2 = 2.5193, p_3 = 2.77623, p_4 = 2.9579, p_5 = 1.9395.$$

Ranking all the teachers X_i, $i = 1, 2, 3, 4, 5$, in accordance with the values of p_i, $i = 1, 2, 3, 4, 5$, we have

$$X_4 \succ X_3 \succ X_2 \succ X_1 \succ X_5,$$

then

$$r_4 \succ r_3 \succ r_2 \succ r_1 \succ r_5.$$

Thus, X_4 is still most desirable alternative.

Table 6 Collective trapezoid fuzzy linguistic decision matrix R^* (for TFLPWHA operator)

	G_1	G_2	G_3	G_4
X_1	$[s_{2.39}, s_{3.59}, s_{5.37}, s_{6.38}]$	$[s_{2.25}, s_{3.45}, s_{5.02}, s_{6.15}]$	$[s_{3.33}, s_{4.44}, s_{5.50}, s_{6.83}]$	$[s_{3.29}, s_{4.29}, s_{5.29}, s_{7.00}]$
X_2	$[s_{2.93}, s_{4.42}, s_{5.68}, s_{6.72}]$	$[s_{1.94}, s_{3.33}, s_{4.54}, s_{5.75}]$	$[s_{4.00}, s_{5.18}, s_{7.12}, s_{8.03}]$	$[s_{3.17}, s_{4.33}, s_{6.00}, s_{7.00}]$
X_3	$[s_{3.05}, s_{4.47}, s_{5.87}, s_{7.62}]$	$[s_{3.03}, s_{4.63}, s_{5.64}, s_{7.00}]$	$[s_{1.86}, s_{4.10}, s_{5.42}, s_{6.51}]$	$[s_{3.11}, s_{4.11}, s_{5.11}, s_{6.11}]$
X_4	$[s_{2.50}, s_{4.93}, s_{6.57}, s_{7.95}]$	$[s_{3.08}, s_{4.64}, s_{6.05}, s_{7.09}]$	$[s_{2.66}, s_{4.12}, s_{5.17}, s_{6.20}]$	$[s_{3.82}, s_{4.82}, s_{6.19}, s_{7.19}]$
X_5	$[s_{1.82}, s_{3.65}, s_{5.00}, s_{6.26}]$	$[s_{1.24}, s_{2.34}, s_{4.18}, s_{6.15}]$	$[s_{2.56}, s_{4.38}, s_{5.40}, s_{6.42}]$	$[s_{3.73}, s_{4.29}, s_{5.80}, s_{7.31}]$

Conclusions

In this paper, we explored multiple attribute group decision-making problems in which the attribute and decision makers are at different priority levels, and the decision information provided by decision makers takes the form of trapezoid linguistic variables. Motivated by the idea of prioritized weighted aggregation operators [42,43], we have developed some prioritized weighted aggregation operators for aggregating trapezoid fuzzy linguistic information: the TFLPWA operator, the TFLPWG operator and the TFLPWHA operator. A number of properties of the proposed operators have been proved. Then, we have developed an algorithm to solve the trapezoid fuzzy linguistic multiple attribute decision-making problems in which the attributes and decision makers are in different priority levels. Finally, a numerical example is given to verify the developed approaches and to demonstrate their practicality and effectiveness.

Received: 13 January 2014 Accepted: 24 March 2014
Published: 28 April 2014

References

1. Beliakov, G, Pradera, A, Calvo, T: Aggregation functions: a guide for practitioners. Springer, Berlin (2007)
2. Calvo, T, Mayor, G, Mesiar, R: Aggregation operators: new trends and applications. Physica-Verlag, Berlin (2002)
3. Xu, ZS: Da, QL: An overview of operators for aggregating information. Int. J. Intell. Syst. **18**(9), 953–969 (2003)
4. Yager, RR, Kacprzyk, J: The ordered weighted averaging operator: theory and applications. Kluwer, Boston (1997)
5. Yager, RR: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Syst. Man Cybern. B Cybern. **18**(1), 183–190 (1988)
6. Bordonga, G, Fedrizzi, M, Passi, G: A linguistic modelling of consensus in group decision making based on OWA operators. IEEE Trans. Syst. Man Cybern. Syst. Hum **27**(1), 126–132 (1997)
7. Chen, H, Liu, C, Sheng, Z: Induced ordered weighted harmonic averaging (IOWHA) operator and its application to combination forecasting method. Chin. J. Manage. Sci. **12**, 35–40 (2004)
8. Chiclana, F, Herrera, F, Viedma, EH: The ordered weighted geometric operator: properties and applications. In: Proceedings of the 8th International Conference on Information Processing and Management of Uncertainty in Knowledge Based Systems, pp. 985–991. Madrid, Spain (2000)
9. Fodor, J, Marichal, JL, Roubens, M: Characterization of the ordered weighted averaging operators. IEEE Trans. Fuzzy Syst. **3**(2), 236–240 (1995)
10. Merigó, JM: A unified model between the weighted average and the induced OWA operator. Expert Syst. Appl. **8**(9), 11560–11572 (2011)
11. Merigó, JM, Gil-Lafuente, AM: The induced generalized OWA operator. Inf. Sci. **179**(6), 729–741 (2009)
12. Merigó, JM, Casanovas, M: The fuzzy generalized OWA operator and its application in strategic decision making. Cybern. Syst. **41**(5), 359–370 (2010)
13. Merigó, JM, Casanovas, M: Induced and uncertain heavy OWA operators. Comp. Ind. Eng. **60**(1), 106–116 (2011)
14. Merigó, JM, Casanovas, M: The uncertain induced quasi-arithmetic OWA operator. Int. J. Intell. Syst. **26**(1), 1–24 (2011)
15. Merigó, JM, Casanovas, M, Martìnez, L: Linguistic aggregation operators for linguistic decision making based on the Dempster-Shafer theory of evidence. International Journal Uncertainty, Fuzziness and Knowledge Based Systems **18**(3), 287–304 (2010)
16. Wang, XR, Fan, ZP: Fuzzy ordered weighted averaging operator and its applications. Fuzzy Syst. Math. **17**(4), 67–72 (2003)
17. Wei, GW: FIOWHM operator and its application to multiple attribute group decision making. Expert Syst. Appl. **38**(4), 2984–2989 (2011)
18. Xu, ZS: Fuzzy ordered weighted geometric operator and its applications in fuzzy AHP. Syst. Eng. Electron. **31**(4), 855–858 (2002)
19. Xu, ZS, QL, D: The ordered weighted geometric averaging operators. Int. J. Intell. Syst. **17**(7), 709–716 (2002)
20. Xu, ZS: A priority method for triangular fuzzy number complementary judgment matrix. Syst. Eng. Theor. Pract. **23**(10), 86–89 (2003)
21. Xu, ZS: EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations. Int. J. Uncertainty Fuzziness Knowl. Based Syst. **12**(6), 791–810 (2004)
22. Xu, ZS: An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. Decis. Support. Syst. **41**(2), 488–499 (2006)
23. Xu, ZS, Xia, G, Chen, M, Wang, L: Induced generalized intuitionistic fuzzy OWA operator for multi-attribute group decision making. Expert Syst. Appl. **39**(2), 1902–1910 (2012)
24. Yager, RR: Families of OWA operators. Fuzzy Sets Syst. **59**(2), 125–148 (1993)
25. Yager, RR: Quantifier guided aggregation using OWA operators. Int. J. Intell. Syst. **11**(1), 49–73 (1996)
26. Yager, RR: Generalized OWA aggregation operators. Fuzzy Optim. Decis. Making **3**(1), 93–107 (2004)
27. Yager, RR: Centered OWA operators. Soft. Comput. **11**(7), 631–639 (2007)
28. Yager, RR, Kacprzyk, J, Beliakov, G: Recent developments in the ordered weighted averaging operator: theory and practice. Springer, Heidelberg (2011)
29. Dyckhoff, H, Pedrycz, W: Generalized means as model of compensative connectives. Fuzzy Sets Syst. **14**(2), 143–154 (1984)

30. Herrera, F, Martínez, L: An approach for combining numerical and linguistic information based on the 2-tuple fuzzy linguistic representation model in decision making. Int.J. Uncertainty Fuzziness Knowl. Based Syst. **8**(5), 539–562 (2000)
31. Xu, ZS: Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Inf. Sci. **168**(1–4), 171–184 (2004)
32. Wei, GW: Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment. Int. J. Uncertainty Fuzziness Knowl. Based Syst. **17**(2), 251–267 (2009)
33. Xu, ZS: Group decision making with triangular fuzzy linguistic variables, pp. 17–26. Springer, Berlin, Heidelberg (2007)
34. Xu, ZS: An approach based on similarity measure to multiple attribute decision making with trapezoid fuzzy linguistic variables, pp. 110–117. Springer, Berlin, Heidelberg (2005)
35. Liang, XC, Chen, SF: Multiple attribute decision making method based on trapezoid fuzzy linguistic variables. J. Southeast Univ. (English Edition) **24**(4), 478–481 (2008)
36. Wei, GW, Yi, WD: Trapezoid fuzzy linguistic weighted geometric averaging operator. In: Proceedings of the 4th International Conference on Fuzzy System and Knowledge Discovery, pp. 300–304. Haikou (2007)
37. Liu, P, Su, Y: The multiple attribute decision making method based on the TFLHOWA operator. Comput. Math. Appl. **60**(9), 2609–2615 (2010)
38. Liu, P, Su, Y: Multiple attribute decision making method based on the trapezoid fuzzy linguistic hybrid harmonic averaging operator. Informatica **36**(1), 83–90 (2012)
39. Bonferroni, C: Sulle medie multiple di potenze. Bollettino dell'Unione Matematica Italiana **5**(3), 267–270 (1950)
40. Liu, P, Jin, Y: The trapezoid fuzzy linguistic Bonferroni mean operators and their application to multiple attribute decision making. Scientia Iranica **19**(6), 1947–1959 (2012)
41. Liu, P, Wu, X: Multiple attribute decision making method based on the generalized aggregation operators in trapezoid fuzzy linguistic variables. J. Comput. Anal. Appl. **15**(5), 807–816 (2013)
42. Yager, RR: Prioritized aggregation operators. Int. J. Approx. Reason. **48**(1), 263–274 (2008)
43. Yager, RR: Prioritized OWA aggregation. Fuzzy Optim. Decis. Making **8**(3), 245–262 (2009)

doi:10.1186/2195-5468-2-10
Cite this article as: Verma and Sharma: **Trapezoid fuzzy linguistic prioritized weighted average operators and their application to multiple attribute group decision-making.** *Journal of Uncertainty Analysis and Applications* 2014 **2**:10.