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Multifactor uncertain differential equation

Shengguo Li[1], Jin Peng[1*] and Bo Zhang[2]

*Correspondence:
pengjin01@tsinghua.org.cn
[1] Institute of Uncertain Systems, Huanggang Normal University, Hubei 438000, China
Full list of author information is available at the end of the article

Abstract
This paper proposes a type of multifactor uncertain differential equation within the framework of uncertainty theory. The analytic solutions of four special types of multifactor uncertain differential equations are first discussed. Then, a numerical method for solving general multifactor uncertain differential equation is presented. Finally, under the Lipschitz condition and linear growth condition, it is proved that the multifactor uncertain differential equation has a unique solution.

Keywords: Uncertainty theory; Canonical process; Uncertainty differential equation; Numerical method

Introduction

Uncertainty theory is a tool to study the indeterminacy phenomena in human systems, which was founded by Liu [1] in 2007. It was refined by Liu [2] and has become an axiomatic system via normality, duality, subadditivity, and product axioms of uncertain measure. Up to now, many branches of mathematics emerged based on uncertainty theory, such as mathematical programming [3], uncertain set and uncertain inference [4], uncertain logic [5], uncertain risk [6,7], and uncertain insurance [8].

Uncertain process is essentially a sequence of uncertain variables indexed by time which was first introduced by Liu [9]. After that, a significant uncertain process called canonical process was designed by [10]. The canonical process is a stationary independent increment process with Lipschitz continuous sample paths. Meanwhile, uncertain calculus with respect to canonical process called Liu calculus was developed by Liu [10]. In order to describe the evolution of uncertain phenomenon with some jumps, Liu [9] proposed the uncertain renewal process. Afterward, Yao [11] presented the uncertain calculus with respect to renewal process called the Yao calculus. Recently, Yao [12] proposed multi-dimensional uncertain calculus with Liu process, Chen [13] studied the uncertain calculus with finite variation processes. More research about uncertain process can be found in references [14-16].

Uncertain differential equation was proposed by Liu [9], which is an important tool to deal with uncertain dynamic systems. Different from stochastic differential equation driven by a Wiener process [17], uncertain differential equation is a type of differential equation driven by uncertain process. In order to know well uncertain differential equation, many researchers did a lot of work. Chen and Liu [18] proved an existence and uniqueness theorem of solution under global Lipschitz condition and proposed an analytic solution for linear uncertain differential equation. Gao [19] gave an existence and uniqueness theorem with local Lipschitz condition. In 2009, Liu [10] gave a concept of stability of uncertain differential equation. After that, Yao et al. [20] proved some stability theorems of uncertain differential equation. In addition, Sheng and Wang [21] investigated the stability in pth moment for uncertain differential equation, Liu et al. [22] studied the almost sure stability, and Yao et al. [23] showed the stability in mean. In order to obtain the solution of uncertain differential equation, Liu [24] and Yao [25] provided the analytic solutions for some special nonlinear uncertain differential equations, respectively. Yao and Chen [26] presented a numerical method for solving uncertain differential equation when it is difficult to obtain analytic solution. Yao [27] also discussed the extreme values and integral of solution of uncertain differential equation.

Uncertain differential equation was first applied in finance by Liu [10] in 2009. Meanwhile, Liu [10] presented an uncertain stock model in uncertain financial market and proved the European option pricing formulas. After that, Chen [28] gave the America option pricing formulas. Besides, Peng and Yao [29] presented another uncertain stock model and corresponding option pricing formulas. Liu [30] discussed some possible applications of uncertain differential equations to financial markets. Li and Peng [31] proposed a stock model with uncertain stock diffusion. Liu et al. [32] built an uncertain currency model and proved the currency option pricing. Jiao and Yao [33] considered an interest rate model in uncertain environment. Yao [34] proved a no-arbitrage theorem for uncertain stock model. In addition, uncertain differential equation was also applied in uncertain optimal control [35] and uncertain differential game [36].

The extensions of uncertain differential equation also attracted the attention of scholars. Several recent contributions in the extension literature have studied this question in many directions. Yao [11] suggested the uncertain differential equation with jumps. Ge and Zhu [37] discussed the backward uncertain differential equation. Barbacioru [38], Ge and Zhu [39], and Liu and Fei [40] focused on the uncertain delay differential equation. Yao [12] proposed the multidimentional uncertain differential equation via multidimensional uncertain calculus. Ji and Zhou [41] proved an existence and uniqueness theorem of solution for multidimensional uncertain differential equation. Yao [42] studied the higher order uncertain differential equation.

Usually, the uncertain factor influencing dynamic systems is not alone. In 2012, Liu and Yao [43] extended uncertain integral from single canonical process to multiple ones. This provides a motivation to consider the concept of uncertain differential equation driven by multiple uncertain processes. In this paper, we present a type of uncertain differential equation driven by multiple canonical processes which can be regarded as a generalization of the uncertain differential equation proposed by Liu [9].

The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory and uncertain calculus are recalled in the 'Preliminary' section. After that, the multifactor uncertain differential equation is presented. Following that, a numerical method is introduced. In addition, an existence and uniqueness theorem is proved. Finally, a brief summary is given.

Preliminary

In this section, uncertainty theory and uncertain calculus are introduced and some basic concepts are given.

Uncertainty theory

Let Γ be a nonempty set and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to $[0, 1]$ ia called uncertain measure if it satisfies the following axioms:

(1) (Normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
(2) (Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
(3) (Subadditivity axiom) for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have:

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertain space. In order to obtain an uncertain measure of compound event, Liu [10] defined a product uncertain measure which produces the fourth axiom of uncertainty theory:

(4) (Product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertain spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ satisfying:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \min_{1 \leq k \leq \infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable is defined as a measurable function from an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set:

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined by Liu [1] as:

$$\Phi(x) = \mathcal{M}\left\{\gamma \in \Gamma \;\middle|\; \xi(\gamma) \leq x\right\},$$

and the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.

An uncertain variable ξ is called normal if it has a normal uncertainty distribution:

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$.

The expected value of uncertain variable ξ is defined by Liu [1] as:

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. The variance of ξ is defined as $V[\xi] = E[(\xi - E[\xi])^2]$.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Liu [2] proved that if $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution:

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Furthermore, the expected value of uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ was obtained by Liu and Ha [44] as follows:

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha.$$

Uncertain calculus

Definition 1. *(Liu [9]) Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set:*

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\}$$

is an event.

Definition 2. *(Liu [10]) An uncertain process C_t is said to be a canonical process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(ii) C_t has stationary and independent increments;
(iii) every increment $C_{s+t} - C_t$ is a normal uncertain variable with expected value 0 and variance t^2.

Definition 3. *(Liu [10]) Let X_t be an uncertain process and C_t be a canonical process. For any partition of closed integral $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as:*

$$\Delta = \max_{1 \le i \le k} \mid t_{i+1} - t_i \mid .$$

Then, Liu integral of X_t with respect to C_t is:

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Example 1. *Let $f(t)$ be a continuous function with respect to t. Then, the uncertain integral:*

$$\int_0^s f(t) dC_t$$

is a normal uncertain variable at each time s, and:

$$\int_0^s f(t) dC_t \sim \mathcal{N}\left(0, \int_0^s \mid f(t) \mid dt\right).$$

Definition 4. *(Liu [9]) Suppose C_t is a canonical process, and f, g are some given functions. Then,*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{1}$$

is called an uncertain differential equation.

The uncertain differential with respect to canonical processes $C_{1t}, C_{2t}, \cdots, C_{nt}$ is defined by Liu and Yao [43] as follows.

Definition 5. *(Liu and Yao [43]) Let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be canonical processes and let Z_t be an uncertain process. If there exist uncertain processes μ_t and $\sigma_{1t}, \sigma_{2t}, \cdots, \sigma_{nt}$ such that:*

$$Z_t = Z_0 + \int_0^t \mu_s ds + \sum_{i=1}^{n} \int_0^t \sigma_{is} dC_{is} \tag{2}$$

for any $t \geq 0$, then, we say Z_t has an uncertain differential:

$$dZ_t = \mu_t dt + \sum_{i=1}^{n} \sigma_{it} dC_{it}. \tag{3}$$

In this case, Z_t is called a differentiable uncertain process with drift μ_t and diffusions $\sigma_{1t}, \sigma_{2t}, \cdots, \sigma_{nt}$.

Theorem 1. *(Liu and Yao [43]) (Fundamental Theorem of Uncertain Calculus) Let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be canonical processes. If $h(t, c_1, c_2, \cdots, c_n)$ is a continuously differentiable function, then the uncertain process $Z_t = h(t, C_{1t}, C_{2t}, \cdots, C_{nt})$ is differentiable and has an uncertain differential:*

$$dZ_t = \frac{\partial h}{\partial t}(t, C_{1t}, C_{2t}, \cdots, C_{nt})dt + \sum_{i=1}^{n} \frac{\partial h}{\partial c_i}(t, C_{1t}, C_{2t}, \cdots, C_{nt})dC_{it}.$$

Multifactor uncertain differential equation

Usually, the uncertain factor influencing dynamic systems is not alone. In order to model the dynamic systems with multiple factors, this section will extend the uncertain differential equation driven by single canonical process to one driven by multiple independent canonical processes.

Definition 6. *(Liu [45]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_1, \cdots, t_k$, the uncertain vectors:*

$$\boldsymbol{\xi}_i = \left(X_{it_1}, X_{it_2}, \cdots, X_{it_k}\right), i = 1, 2, \cdots, n$$

are independent, i.e., for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have:

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}.$$

Theorem 2. *(Liu [45]) Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then:*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt})$$

is an uncertain variable with inverse uncertainty distribution:

$$\Phi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha), \cdots, \Phi_{nt}^{-1}(1-\alpha)).$$

Definition 7. *Suppose $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent canonical processes, and f, $g_1, g_2, \cdots, g_n$ are some given functions. Then:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} g_i(t, X_t)dC_{it} \tag{4}$$

is called an uncertain differential equation with respect to $C_{1t}, C_{2t}, \cdots, C_{nt}$. A solution is an uncertain process X_t that satisfies Equation 4 identically in t.

The uncertain differential Equation 4 is equivalent to the uncertain integral equation:

$$X_s = X_0 + \int_0^s f(t, X_t)dt + \sum_{i=1}^{n} \int_0^s g_i(t, X_t)dC_{it}. \tag{5}$$

Example 2. *Let a, b and c be real numbers, and let C_{1t}, C_{2t} be independent canonical processes. The uncertain differential equation:*

$$dX_t = adt + bdC_{1t} + cdC_{2t} \tag{6}$$

has a solution:

$$X_t = X_0 + at + bC_{1t} + cC_{2t}. \tag{7}$$

Theorem 3. *Let μ_t, $\nu_{1t}, \nu_{2t}, \cdots, \nu_{nt}$ be integrable uncertain processes and let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be independent canonical processes. Then, the uncertain differential equation:*

$$dX_t = \mu_t X_t dt + \sum_{i=1}^{n} \nu_{it} X_t dC_{it} \tag{8}$$

has a solution:

$$X_t = X_0 \exp\left(\int_0^t \mu_s ds + \sum_{i=1}^{n} \int_0^t \nu_{is} dC_{is}\right). \tag{9}$$

Proof. At first, the original uncertain differential equation is equivalent to:

$$\frac{dX_t}{X_t} = \mu_t dt + \sum_{i=1}^{n} \nu_{it} dC_{it}.$$

It follows from the fundamental theorem of uncertain calculus that:

$$d\ln X_t = \frac{dX_t}{X_t} = \mu_t dt + \sum_{i=1}^{n} \nu_{it} dC_{it}$$

and then:

$$\ln X_t = \ln X_0 + \int_0^t \mu_s ds + \sum_{i=1}^{n} \int_0^t \nu_{is} dC_{is}.$$

Therefore, the uncertain differential Equation 8 has a solution (9).

Example 3. *Let a, b, and c be real numbers, and let* C_{1t} *and* C_{2t} *be independent canonical processes. The uncertain differential equation:*

$$dX_t = aX_t dt + bX_t dC_{1t} + cX_t dC_{2t} \tag{10}$$

has a solution:

$$X_t = X_0 \exp\left(at + bC_{1t} + cC_{2t}\right). \tag{11}$$

Theorem 4. *Let* μ_{1t}*,* μ_{2t}*,* $\nu_{1t}, \nu_{2t}, \cdots, \nu_{nt}$ *and* $\omega_{1t}, \omega_{2t}, \cdots, \omega_{nt}$ *be integrable uncertain processes. Assume* $C_{1t}, C_{2t}, \cdots, C_{nt}$ *are independent canonical processes, then the uncertain differential equation:*

$$dX_t = (\mu_{1t}X_t + \mu_{2t})dt + \sum_{i=1}^{n}(\nu_{it}X_t + \omega_{it})dC_{it} \tag{12}$$

has a solution:

$$X_t = U_t\left(X_0 + \int_0^t \frac{\mu_{2s}}{U_s}ds + \sum_{i=1}^{n}\int_0^t \frac{\omega_{is}}{U_s}dC_{is}\right) \tag{13}$$

where:

$$U_t = \exp\left(\int_0^t \mu_{1s}ds + \sum_{i=1}^{n}\int_0^t \nu_{is}dC_{is}\right). \tag{14}$$

Proof. Define two uncertain processes U_t and V_t via uncertain differential equations,

$$dU_t = \mu_t U_t dt + \sum_{i=1}^{n}\nu_{it}U_t dC_{it},$$

$$dV_t = \frac{\mu_{2t}}{U_t}dt + \sum_{i=1}^{n}\frac{\omega_{it}}{U_t}dC_{it}.$$

It follows from the integration by parts that:

$$d(U_tV_t) = V_t dU_t + U_t dV_t = (\mu_{1t}U_tV_t + \mu_{2t}dt) + \sum_{i=1}^{n}(\nu_{it}U_tV_t + \omega_{it})C_{it}.$$

That is, the uncertain process $X_t = U_tV_t$ is a solution of the uncertain differential Equation (12). Note that:

$$U_t = U_0 \exp\left(\int_0^t \mu_{1s}ds + \sum_{i=1}^{n}\int_0^t \nu_{is}dC_{is}\right),$$

$$V_t = V_0 + \int_0^t \frac{\mu_{2s}}{U_s}dt + \sum_{i=1}^{n}\int_0^t \frac{\omega_{is}}{U_s}dC_{is}.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we get the solutions (13) and (14). The theorem is proved.

Note that $n = 1$, the uncertain differential Equation 12 degenerates to the linear uncertain differential equation in Chen and Liu [18].

Example 4. *Let m, a,* σ*, and* ω *be real numbers and let* C_{1t} *and* C_{2t} *be independent canonical processes. The uncertain differential equation:*

$$dX_t = (m - aX_t)dt + \sigma dC_{1t} + \omega dC_{2t} \tag{15}$$

has the solution:

$$X_t = \exp(-at)\left(X_0 + \frac{m}{a}(\exp(at)-1) + \int_0^t \sigma \exp(as)dC_{1s} + \int_0^t \omega \exp(as)dC_{2s}\right) \quad (16)$$

provided that $a \neq 0$.

Example 5. *Let* m, σ, *and* ω *be real numbers and let* C_{1t} *and* C_{2t} *be independent canonical processes. The uncertain differential equation:*

$$dX_t = mdt + \sigma X_t dC_{1t} + \omega X_t dC_{2t} \quad (17)$$

has the solution:

$$X_t = \exp(\sigma C_{1t} + \omega C_{2t})\left(X_0 + \int_0^t m\exp(\sigma C_{1s} + \omega C_{2s})ds\right). \quad (18)$$

Theorem 5. *Let* f *be a function of two variables and let* $\sigma_{1t}, \sigma_{2t}, \cdots, \sigma_{nt}$ *be integrable uncertain processes. Assume* $C_{1t}, C_{2t}, \cdots, C_{nt}$ *are independent canonical processes, then the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} \sigma_{it} X_t dC_{it} \quad (19)$$

has a solution:

$$X_t = Y_t^{-1} Z_t \quad (20)$$

where:

$$Y_t = \exp\left(-\sum_{i=1}^{n}\int_0^t \sigma_{is} dC_{is}\right) \quad (21)$$

and Z_t *is the solution of uncertain differential equation:*

$$dZ_t = Y_t f\left(t, Y_t^{-1} Z_t\right) dt \quad (22)$$

with initial value $Z_0 = X_0$.

Proof. By the fundamental theorem of uncertain calculus, the uncertain process Y_t has an uncertain differential:

$$dY_t = -\exp\left(-\sum_{i=1}^{n}\int_0^t \sigma_{is} dC_{is}\right)\sum_{i=1}^{n}\sigma_{it} dC_{it} = -Y_t \sum_{i=1}^{n}\sigma_{it} dC_{it}.$$

It follows from the integration by parts that:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t = -X_t Y_t \sum_{i=1}^{n}\sigma_{it} dC_{it} + Y_t f(t, X_t) + X_t Y_t \sum_{i=1}^{n}\sigma_{it} dC_{it}.$$

That is,

$$d(X_t Y_t) = Y_t f(t, X_t).$$

Defining $Z_t = X_t Y_t$, we obtain $X_t = Y_t^{-1} Z_t$ and $dZ_t = Y_t f\left(t, Y_t^{-1} Z_t\right)$. Furthermore, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is proved.

Note that $n = 1$, the uncertain differential Equation 19 degenerates to the nonlinear uncertain differential equation in Liu [24].

Example 6. *Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be real numbers and let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be independent canonical processes. Consider the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} \sigma_i X_t dC_{it}. \tag{23}$$

Theorem 5 shows that:

$$Y_t = \exp\left(-\sum_{i=1}^{n} \sigma_i C_{is}\right)$$

and:

$$X_t = \exp\left(\sum_{i=1}^{n} \sigma_i C_{is}\right) Z_t$$

where Z_t is the solution of uncertain differential equation:

$$dZ_t = \exp\left(-\sum_{i=1}^{n} \sigma_i C_{is}\right) f\left(t, \exp\left(\sum_{i=1}^{n} \sigma_i C_{is}\right) Z_t\right) dt$$

with initial value $Z_0 = X_0$. Taking $f(t, X_t) = X_t^{\alpha}$, $\alpha \neq 1$, we can obtain:

$$dZ_t^{1-\alpha} = (1-\alpha) \exp\left((1-\alpha) \sum_{i=1}^{n} \sigma_i C_{it}\right) dt$$

and:

$$X_t = \exp\left(\sum_{i=1}^{n} \sigma_i C_{it}\right) \left(X_0^{1-\alpha} + (1-\alpha) \int_0^t \exp\left((1-\alpha) \sum_{i=1}^{n} \sigma_i C_{is}\right) ds\right)^{\frac{1}{1-\alpha}}.$$

Theorem 6. *Let $g_1, g_2, \cdots, g_n$ be functions of two variables and let α_t be an integrable uncertain process. Assume $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent canonical processes, then the uncertain differential equation:*

$$dX_t = \alpha_t X_t dt + \sum_{i=1}^{n} g_i(t, X_t) dC_{it} \tag{24}$$

has a solution:

$$X_t = Y_t^{-1} Z_t \tag{25}$$

where:

$$Y_t = \exp\left(-\int_0^t \alpha_s ds\right) \tag{26}$$

and Z_t is the solution of uncertain differential equation:

$$dZ_t = Y_t \sum_{i=1}^{n} g_i\left(t, Y_t^{-1} Z_t\right) dt \tag{27}$$

with initial value $Z_0 = X_0$.

Proof. It follows from the fundamental theorem of uncertain calculus that:

$$dY_t = -\exp\left(-\int_0^t \alpha_s ds\right) \alpha_t dt = -Y_t \alpha_t dt.$$

Using the integration by parts, we have the following:

$$d(X_tY_t) = X_tdY_t + Y_tdX_t = -X_tY_t\alpha_tdt + Y_t\alpha_tX_tdt + Y_t\sum_{i=1}^{n} g_i(t, X_t)dC_{it}.$$

That is,

$$d(X_tY_t) = Y_t\sum_{i=1}^{n} g_i(t, X_t)dC_{it}.$$

Define $Z_t = X_tY_t$, then $X_t = Y_t^{-1}Z_t$ and $dZ_t = Y_t\sum_{i=1}^{n} g_i\left(t, Y_t^{-1}Z_t\right)dC_{it}$. In addition, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is proved.

Note that $n = 1$, the uncertain differential Equation 24 degenerates to the nonlinear uncertain differential equation in Liu [24].

Example 7. *Let α , b, c, and β be real numbers with $\beta \neq 1$, and let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be independent canonical processes. Consider the uncertain differential equation:*

$$dX_t = \alpha X_tdt + bX_t^{\beta}dC_{1t} + cX_t^{\beta}dC_{2t}. \tag{28}$$

At first,

$$Y_t = \exp(-\alpha t)$$

and Z_t satisfies uncertain differential equation:

$$dZ_t = b\exp((\beta-1)\alpha t)Z_t^{\beta}dC_{1t} + c\exp((\beta-1)\alpha t)Z_t^{\beta}dC_{2t}.$$

Since $\beta \neq 1$, we have:

$$dZ_t^{1-\alpha} = (1-\beta)(b\exp((\beta-1)\alpha t)dC_{1t} + c\exp((\beta-1)\alpha t)dC_{2t}).$$

It follows from the fundamental theorem of uncertain calculus that:

$$Z_t^{1-\alpha} = Z_0^{1-\alpha} + (1-\beta)\left(b\int_0^t \exp((\beta-1)\alpha s)dC_{1s} + c\int_0^t \exp((\beta-1)\alpha s)dC_{2s}\right).$$

Theorem 6 says the uncertain differential equation has a solution:

$$X_t = \exp(\alpha t)\left(X_0^{1-\alpha} + (1-\beta)(b\int_0^t \exp((\beta-1)\alpha s)dC_{1s} + c\int_0^t \exp((\beta-1)\alpha s)dC_{2s}\right)^{\frac{1}{1-\beta}}.$$

Numerical method

However, in many cases, it is difficult to find analytic solutions of uncertain differential equations. Yao and Chen [26] presented a numerical method called Yao-Chen method to obtain the inverse uncertainty distribution of solution.

Yao-Chen formula

Definition 8. *(Yao and Chen [26]) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{29}$$

is said to have an α-path X_t^{α} if it solves the corresponding ordinary differential equation:

$$dX_t^{\alpha} = f\left(t, X_t^{\alpha}\right)dt + \mid g\left(t, X_t^{\alpha}\right) \mid \Phi^{-1}(\alpha)dt \tag{30}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Theorem 7. *(Yao-Chen Formula [26]) Assume that f, $g_1, g_2, \cdots, g_n$ are continuous functions of two variables. Let X_t and X_t^α be the solution and α-path of the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then:

$$\mathcal{M}\left\{X_t \le X_t^\alpha, \forall t\right\} = \alpha,$$

$$\mathcal{M}\left\{X_t > X_t^\alpha, \forall t\right\} = 1 - \alpha.$$

Theorem 8. *(Yao and Chen [26]) Assume that f, $g_1, g_2, \cdots, g_n$ are continuous functions of two variables. Let X_t and X_t^α be the solution and α-path of the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t,$$

respectively. Then, the solution X_t has an inverse uncertainty distribution:

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Generalization

In this subsection, we generalize the Yao-Chen formula to the multifactor uncertain differential equation.

Definition 9. *Let α be a number with $0 < \alpha < 1$, and let $C_{1t}, C_{2t}, \cdots, C_{nt}$ be independent canonical processes. An uncertain differential equation:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} g_i(t, X_t)dC_{it} \tag{31}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation:

$$dX_t^\alpha = f\left(t, X_t^\alpha\right)dt + \sum_{i=1}^{n} \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha)dt \tag{32}$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable $\mathcal{N}(0, 1)$, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \ 0 < \alpha < 1.$$

Example 8. *Let a, b, and c be real numbers. The uncertain differential equation:*

$$dX_t = adt + bdC_{1t} + cdC_{2t}, \ X_0 = 0$$

has an α-path:

$$X_t^\alpha = at + (\mid b \mid + \mid c \mid)\Phi^{-1}(\alpha).$$

Lemma 9. *Assume that $f(t,x)$ and $g(t,x)$ are continuous functions. Let $\phi(t)$ be a solution of the ordinary differential equation:*

$$\frac{dx}{dt} = f(t,x)dt + K \mid g(t,x) \mid, \; x(0) = x_0$$

where K is a real number. Let $\psi(t)$ be a solution of the ordinary differential equation:

$$\frac{dx}{dt} = f(t,x)dt + k(t)g(t,x), \; x(0) = x_0$$

where $k(t)$ is a real function.

(i) *If $k(t)g(t,x) \leq K \mid g(t,x) \mid$ for $t \in [0,T]$, then $\psi(T) \leq \phi(T)$,*
(ii) *If $k(t)g(t,x) > K \mid g(t,x) \mid$ for $t \in [0,T]$, then $\psi(T) > \phi(T)$.*

Theorem 10. *Assume that $f, g_1, g_2, \cdots, g_n$ are continuous functions of two variables and $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent canonical processes. Let X_t and X_t^α be the solution and α-path of the uncertain differential equation:*

$$dX_t = f(t,X_t)dt + \sum_{i=1}^{n} g_i(t,X_t)dC_{it},$$

respectively. Then:

$$\mathcal{M}\left\{X_t \leq X_t^\alpha, \forall t\right\} = \alpha,$$

$$\mathcal{M}\left\{X_t > X_t^\alpha, \forall t\right\} = 1-\alpha.$$

Proof. For each α-path X_t^α, we construct sets as follows,

$$T_i^+ = \left\{t \mid g_i\left(t, X_t^\alpha\right) \geq 0\right\},$$

$$T_i^- = \left\{t \mid g_i\left(t, X_t^\alpha\right) < 0\right\},$$

$i = 1, 2, \cdots, n$. It is obvious that $T_i^+ \cap T_i^- = \emptyset$ and $T_i^+ \cup T_i^- = [0, +\infty)$ for each $1 \leq i \leq n$.
Write:

$$\Lambda_{i1}^+ = \left\{\gamma \mid \frac{dC_{it}(\gamma)}{dt} \leq \Phi^{-1}(\alpha) \text{ for } t \in T_i^+\right\},$$

$$\Lambda_{i1}^- = \left\{\gamma \mid \frac{dC_{it}(\gamma)}{dt} \geq \Phi^{-1}(1-\alpha) \text{ for } t \in T_i^-\right\},$$

$i = 1, 2, \cdots, n$, where Φ^{-1} is the inverse uncertainty distribution of $\mathcal{N}(0,1)$. Since T_i^+ and T_i^- are disjoint sets and C_{it} have independent increments, we get:

$$\mathcal{M}\left\{\Lambda_{i1}^+\right\} = \alpha, \quad \mathcal{M}\left\{\Lambda_{i1}^-\right\} = \alpha \quad \mathcal{M}\left\{\Lambda_{i1}^+ \cap \Lambda_{i1}^-\right\} = \alpha.$$

For any $\gamma \in \Lambda_{i1}^+ \cap \Lambda_{i1}^-$, we always have:

$$g_i(t, X_t(\gamma))\frac{dC_{it}(\gamma)}{dt} \leq \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha), \forall t, \; i = 1, 2, \cdots, n.$$

Let $\Lambda_1^+ \cap \Lambda_1^- = \bigcap_{i=1}^{n} \left(\Lambda_{i1}^+ \cap \Lambda_{i1}^-\right)$. Because $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent and $\mathcal{M}\left\{\Lambda_{i1}^+ \cap \Lambda_{i1}^-\right\} = \alpha$, $i = 1, 2, \cdots, n$, we have:

$$\mathcal{M}\left\{\Lambda_1^+ \cap \Lambda_1^-\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}\left(\Lambda_{i1}^+ \cap \Lambda_{i1}^-\right)\right\} = \min_{1\le i\le n} \mathcal{M}\left\{\Lambda_{i1}^+ \cap \Lambda_{i1}^-\right\} = \alpha.$$

Then, for any $\gamma \in \Lambda_1^+ \cap \Lambda_1^-$, we have:

$$\sum_{i=1}^{n} g_i(t, X_t(\gamma)) \frac{dC_{it}(\gamma)}{dt} \le \sum_{i=1}^{n} \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha), \forall t.$$

The Lemma 9 shows that $X_t \le X_t^\alpha$ for all t, so $\Lambda_1^+ \cap \Lambda_1^- \subset \left\{X_t \le X_t^\alpha, \forall t\right\}$. Hence:

$$\mathcal{M}\left\{X_t \le X_t^\alpha, \forall t\right\} \ge \mathcal{M}\left\{\Lambda_1^+ \cap \Lambda_1^-\right\} = \alpha. \tag{33}$$

On the other hand, write:

$$\Lambda_{i2}^+ = \left\{\gamma \mid \frac{dC_{it}(\gamma)}{dt} > \Phi^{-1}(\alpha) \text{ for } t \in T_i^+\right\},$$

$$\Lambda_{i2}^- = \left\{\gamma \mid \frac{dC_{it}(\gamma)}{dt} < \Phi^{-1}(1-\alpha) \text{ for } t \in T_i^-\right\},$$

$i = 1, 2, \cdots, n$. Since T_i^+ and T_i^- are disjoint sets and C_{it} has independent increments, we get:

$$\mathcal{M}\left\{\Lambda_{i2}^+\right\} = 1-\alpha, \quad \mathcal{M}\left\{\Lambda_{i2}^-\right\} = 1-\alpha, \quad \mathcal{M}\left\{\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right\} = 1-\alpha.$$

For any $\gamma \in \Lambda_{i2}^+ \cap \Lambda_{i2}^-$, we always have:

$$g_i(t, X_t(\gamma)) \frac{dC_{it}(\gamma)}{dt} > \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha), \forall t, \; i = 1, 2, \cdots, n.$$

Let $\Lambda_2^+ \cap \Lambda_2^- = \bigcap_{i=1}^{n} \left(\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right)$. Because $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent and $\mathcal{M}\left\{\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right\} = 1-\alpha$, $i = 1, 2, \cdots, n$, we have:

$$\mathcal{M}\left\{\Lambda_2^+ \cap \Lambda_2^-\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}\left(\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right)\right\} = \min_{1\le i\le n} \mathcal{M}\left\{\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right\} = 1-\alpha.$$

Then, for any $\gamma \in \Lambda_2^+ \cap \Lambda_2^-$, we have:

$$\sum_{i=1}^{n} g_i(t, X_t(\gamma)) \frac{dC_{it}(\gamma)}{dt} > \sum_{i=1}^{n} \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha), \forall t.$$

The Lemma 9 shows that $X_t > X_t^\alpha$ for any t, so $\Lambda_2^+ \cap \Lambda_2^- \subset \left\{X_t > X_t^\alpha, \forall t\right\}$. Hence:

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \ge \mathcal{M}\left\{\Lambda_{i2}^+ \cap \Lambda_{i2}^-\right\} = 1-\alpha. \tag{34}$$

Since $\left\{X_t \le X_t^\alpha, \forall t\right\}$ and $\left\{X_t \not\le X_t^\alpha, \forall t\right\}$ are opposite events with each other. It follows from the duality axiom that:

$$\mathcal{M}\left\{X_t \le X_t^\alpha, \forall t\right\} + \mathcal{M}\left\{X_t \not\le X_t^\alpha, \forall t\right\} = 1.$$

In addition, $\left\{X_t > X_t^\alpha, \forall t\right\} \subset \left\{X_t \not\le X_t^\alpha, \forall t\right\}$ means that:

$$\mathcal{M}\left\{X_t \le X_t^\alpha, \forall t\right\} + \mathcal{M}\left\{X_t > X_t^\alpha, \forall t\right\} \le 1. \tag{35}$$

Thus, the results follow from (33), (34), and (35).

Theorem 11. *Assume that $f, g_1, g_2, \cdots, g_n$ are continuous functions of two variables and $C_{1t}, C_{2t}, \cdots, C_{nt}$ are independent canonical processes. Let X_t and X_t^α be the solution and α-path of the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} g_i(t, X_t)dC_{it},$$

respectively. Then, the solution X_t has an inverse uncertainty distribution:

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Proof. Obviously, $\left\{X_t \le X_t^\alpha\right\} \supset \left\{X_s \le X_s^\alpha, \forall s\right\}$. It follows from the monotonicity theorem and Theorem 10 that:

$$\mathcal{M}\left\{X_t \le X_t^\alpha\right\} \ge \mathcal{M}\left\{X_s \le X_s^\alpha, \forall s\right\} = \alpha. \tag{36}$$

Similarly, we also obtain:

$$\mathcal{M}\left\{X_t > X_t^\alpha\right\} \ge \mathcal{M}\left\{X_s > X_s^\alpha, \forall s\right\} = 1 - \alpha. \tag{37}$$

Besides, by using the duality axiom, we have:

$$\mathcal{M}\left\{X_t \le X_t^\alpha\right\} + \mathcal{M}\left\{X_t > X_t^\alpha\right\} = 1. \tag{38}$$

It follows from (36), (37), and (38) that:

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Example 9. *Let a, b, and c be real numbers and let C_{1t} and C_{2t} be independent canonical processes. The uncertain differential equation:*

$$dX_t = aX_t dt + bX_t dC_{1t} + cX_t dC_{2t}, \ X_0 = 1 \tag{39}$$

has a solution:

$$X_t = \exp(at + bC_{1t} + cC_{2t})$$

with an inverse uncertainty distribution:

$$\Psi_t^{-1}(\alpha) = \exp\left(at + (\mid b \mid + \mid c \mid)\Phi^{-1}(\alpha)\right).$$

Based on the previous theorem, the Yao-Chen method can be generalized to the multifactor uncertain differential equation as follows.

Step 1: Fix α on $(0, 1)$.

Step 2: Solve the corresponding ordinary differential equation:

$$dX_t^\alpha = f\left(t, X_t^\alpha\right)dt + \sum_{i=1}^{n} \mid g_i\left(t, X_t^\alpha\right) \mid \Phi^{-1}(\alpha)dt$$

and obtain X_t^α, for example, we can choose the recursion formula:

$$X_{i+1}^\alpha = X_i^\alpha + f\left(t_i, X_i^\alpha\right)h + \sum_{j=1}^{n} \mid g_j\left(t_i, X_i^\alpha\right) \mid \Phi^{-1}(\alpha)h$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution and h is the step length.

Step 3: The inverse uncertainty distribution of X_t is obtained.

Example 10. *In order to illustrate the numerical method, let us consider an uncertain differential equation:*

$$\mathrm{d}X_t = X_t\mathrm{d}t + X_t\mathrm{d}C_{1t} + X_t\mathrm{d}C_{2t},\ X_0 = 1 \tag{40}$$

whose solution is $X_t = \exp(t + C_{1t} + C_{2t})$. *The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this equation successfully and obtain an inverse uncertainty distribution of* X_t *at* $t = 1/2$ *shown in Figure 1.*

Existence and uniqueness theorem

This section will give an existence and uniqueness theorem of solution for the multifactor uncertain differential equation under Lipschitz condition and linear growth condition.

Lemma 12. *(Chen and Liu [18]) Let* C_t *be a canonical process, and* X_t *an integrable uncertain process on* $[a, b]$ *with respect to t. Then, the inequality:*

$$\mid \int_a^b X_t(\gamma)dC_t(\gamma) \mid \leq K(\gamma)\int_a^b \mid X_t(\gamma) \mid dt$$

holds, where $K(\gamma)$ *is the Lipschitz constant of the sample path* $X_t(\gamma)$.

Theorem 13. *Let* $f, g_1, g_2, \cdots, g_n$ *be functions of two variables and let* $C_{1t}, C_{2t}, \cdots, C_{nt}$ *be independent canonical processes. Then, the uncertain differential equation:*

$$dX_t = f(t, X_t)dt + \sum_{i=1}^{n} g_i(t, X_t)dC_{it}$$

has a unique solution if the coefficients $f, g_1, g_2, \cdots, g_n$ *satisfy the Lipschitz condition:*

$$\mid f(t,x) - f(t,y) \mid + \sum_{i=1}^{n} \mid g_i(t,x) - g_i(t,y) \mid \leq L \mid x - y \mid, \text{ for all } x, y \in \Re,\ t \geq 0 \tag{41}$$

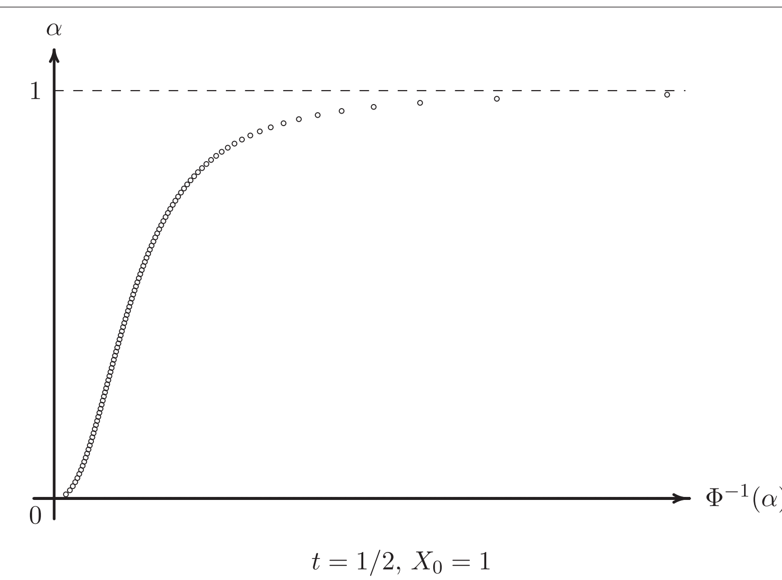

Figure 1 $t = 1/2, X_0 = 1$.

and linear growth condition:

$$| f(t,x) | + \sum_{i=1}^{n} | g_i(t,x) | \leq L(1+ | x |), \text{ for all } x \in \Re,\ t \geq 0 \tag{42}$$

for some constant L. Moreover, the solution is sample-continuous.

Proof. We first prove the existence of solution by a successive approximation method. Define $X_t^{(0)} = X_0$, and:

$$X_t^{(n)} = X_0 + \int_0^t f\left(s, X_s^{(n-1)}\right) ds + \sum_{i=1}^{n} \int_0^t g_i\left(s, X_s^{(n-1)}\right) dC_{is}$$

for $n = 1, 2, \cdots, n$ and write:

$$D_t^n(\gamma) = \max_{0 \leq s \leq t} | X_s^{(n+1)}(\gamma) - X_s^{(n)}(\gamma) |$$

for each $\gamma \in \Gamma$.

We claim that:

$$D_t^n(\gamma) \leq (1+ | X_0 |) \frac{L^{n+1}(1+K_\gamma)^{n+1}}{(n+1)!} t^{n+1}$$

where $K_\gamma = \sum_{i=1}^{n} K_{i\gamma}$, and $K_{i\gamma}$ is the Lipschitz constant to the sample path $C_{it}(\gamma)$, $i = 1, 2, \cdots, n$.

For $n = 0$, we have:

$$\begin{aligned} D_t^{(0)}(\gamma) &= \max_{0 \leq s \leq t} | \int_0^s f(v, X_0) dv + \sum_{i=1}^{n} \int_0^s g_i(v, X_0) dC_{iv}(\gamma) | \\ &\leq \int_0^t | f(v, X_0) | dv + \sum_{i=1}^{n} K_{i\gamma} \int_0^t | g_i(v, X_0) | dv \\ &\leq (1+ | X_0 |) L(1+K_\gamma) t \end{aligned}$$

where the first inequality comes from Lemma 12, the second comes from the linear growth condition.

This confirms the claim for $n = 0$. Next, we assume the claim is true for some $n - 1$. Then:

$$\begin{aligned}
D_t^{(n)}(\gamma) &= \max_{0\le s\le t} \mid \int_0^s (f(v, X_v^{(n)}(\gamma)) - f\left(v, X_v^{(n-1)}(\gamma)\right) dv \\
&\quad + \sum_{i=1}^n \int_0^s \left(g_i(v, X_v^{(n)}(\gamma)\right) - g_i\left(v, X_v^{(n-1)}(\gamma)\right) dC_{iv}(\gamma) \mid \\
&\le \int_0^t \mid \left(f(v, X_v^{(n)}(\gamma)\right) - f\left(v, X_v^{(n-1)}(\gamma)\right) \mid dv \\
&\quad + \sum_{i=1}^n \int_0^t \mid \left(g_i(v, X_v^{(n)}(\gamma)\right) - g_i\left(v, X_v^{(n-1)}(\gamma)\right) \mid dC_{iv}(\gamma) \\
&\le L\int_0^t \mid X_v^{(n)}(\gamma) - X_v^{(n-1)}(\gamma) \mid dv \\
&\quad + \sum_{i=1}^n K_{i\gamma} \int_0^t \mid \left(g_i(v, X_v^{(n)}(\gamma)\right) - g_i\left(v, X_v^{(n-1)}(\gamma)\right) \mid dv \\
&\le L\int_0^t \mid X_v^{(n)}(\gamma) - X_v^{(n-1)}(\gamma) \mid dv \\
&\quad + L\sum_{i=1}^n K_{i\gamma} \int_0^t \mid X_v^{(n)}(\gamma) - X_v^{(n-1)}(\gamma) \mid dv \\
&\le L(1+K_\gamma)\int_0^t \mid X_v^{(n)}(\gamma) - X_v^{(n-1)}(\gamma) \mid dv \\
&\le L(1+K_\gamma)\int_0^t (1+ \mid X_0 \mid) \frac{L^n(1+K_\gamma)^n}{n!} v^n dv \\
&= (1+ \mid X_0 \mid) \frac{L^{n+1}(1+K_\gamma)^{n+1}}{(n+1)!} t^{n+1}.
\end{aligned}$$

It follows from Weierstrassąŕ criterion that, for each sample γ, the paths $X_t^{(k)}(\gamma)$ converges uniformly on any given interval $[0, T]$. Write the limit by $X_t(\gamma)$ that is just a solution:

$$X_t = X_0 + \int_0^t f(s, X_s) ds + \sum_{i=1}^n \int_0^t g_i(s, X_s) dC_{is}.$$

Next, we prove that the solution is unique. Assume that X_t and $X_t^\star$ are solutions. The Lipschitz condition and linear growth condition show:

$$\mid X_t(\gamma) - X_t^\star(\gamma) \mid \le L(1+K_\gamma)\int_0^t \mid X_v(\gamma) - X_v^\star(\gamma) \mid dv.$$

It follows from Gronwall inequality that:

$$\mid X_t(\gamma) - X_t^\star(\gamma) \mid \le 0 \cdot \exp(L(1+K_\gamma)).$$

Hence, $X_t = X_t^\star$. The uniqueness is proved.

At last, we will prove the sample-continuity of X_t. For each $\gamma \in \Gamma$, by the above proof, we get:

$$X_t(\gamma) \le \sum_{n=0}^{+\infty} (1+ \mid X_0 \mid) \frac{L^{n+1}(1+K_\gamma)^{n+1}}{(n+1)!} t^{n+1} = (1+ \mid X_0 \mid) \exp(L(1+K_\gamma)t).$$

Suppose $0 < s < t$, we have:

$$\begin{aligned} | X_t(\gamma) - X_s(\gamma) | &= | \int_s^t f(\upsilon, X_\upsilon) d\upsilon + \sum_{i=1}^n \int_s^t g_i(\upsilon, X_\upsilon) dC_{i\upsilon} | \\ &\le \int_s^t | f(\upsilon, X_\upsilon) | d\upsilon + \sum_{i=1}^n \int_s^t | g_i(\upsilon, X_\upsilon) | dC_{i\upsilon} \\ &\le \int_s^t | f(\upsilon, X_\upsilon) | d\upsilon + \sum_{i=1}^n K_{i\gamma} \int_s^t | g_i(\upsilon, X_\upsilon) | d\upsilon \\ &\le (1 + K_\gamma) L (1 + | X_\upsilon(\gamma) |)(t - s) \\ &\le (1 + K_\gamma) L (1 + (1 + | X_0 |) \exp(L(1 + K_\gamma)t))(t - s). \end{aligned}$$

Thus $| X_t(\gamma) - X_s(\gamma) | \to 0$ as $s \to t$. Hence, X_t is sample-continuous. The theorem is proved.

Note that $n = 1$, the existence and uniqueness theorem degenerates to the one in Chen and Liu [18].

Conclusions

Uncertain differential equation is an important tool to deal with dynamic systems in uncertain environments. In this paper, the multifactor uncertain differential equation was proposed. Four special types of multifactor uncertain differential equations were studied and the corresponding analytic solutions were given. For general multifactor uncertain differential equation, a numerical method was provided for obtaining the solution. Also, an existence and uniqueness theorem that the multifactor uncertain differential equation has a unique solution was proved. The proposed multifactor uncertain differential equation can be used to describe the multifactor stock model in uncertain market.

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Author details
[1]Institute of Uncertain Systems, Huanggang Normal University, Hubei 438000, China. [2]School of Statistics and Mathematics, Zhongnan University of Economics and Law, Hubei 430073, China.

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