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Delphi Method for Estimating Membership Function of Uncertain Set

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Abstract
Uncertain set is a set-valued function on an uncertainty space, and attempts to model unsharp concepts. Membership function is used to describe the membership degree that a value belongs to the uncertain set. In order to estimate the membership function for an uncertain set via multiple experts' experimental data, this paper presents a new method by the combination of the method of uncertain statistics and Delphi method, and gives an example to verify the method.

Keywords: Uncertain set, Membership function, Uncertain statistics, Delphi method

Introduction

In the dance competition, experts give their own judgments such as superior, middle, or inferior according to the dancers' performance. For the examination paper, teachers tend to judge the degrees of papers' difficulty in advance. In fact, when forecasting and decision-making in practice, experts will judge according to their own experience and knowledge. And, experts' advice is often crucial. How to distinguish the experience and knowledge given by experts? How to understand experts' experience and knowledge? How to express in mathematical language? Different scholars have proposed the different approaches. A traditional method is to use probability theory. The experts' advice is characterized by subjective probability. Another method is to use the fuzzy set proposed by Zadeh [1]. Experts' experience and knowledge are interpreted as experts' belief degree by the two methods in common. Each method is constantly challenged since being proposed. It is pointed that human beings usually overweight are unlikely events by Kahneman and Tversky [2], so the belief degree may have much larger variance than the real frequency. Now, if we deal with the belief degree by probability theory, it will lead to making wrong decisions. A specific counterexample supporting this view was presented by Liu [3]. Another example was also presented by Liu [3] saying that the fuzzy set is not suitable for unsharp concepts such as "young".

In order to model human uncertainty well with the axiomatic mathematical model, uncertainty theory was found by Liu [4] in 2007 and refined by Liu [5] in 2010. The basic idea of the uncertainty theory comes from the measure theory in classical mathematics and probability theory. The first fundamental concept in the uncertainty theory is an uncertain measure. It is used to indicate the belief degree that an uncertain event may occur. The second one is uncertain variable that represents quantities in uncertainty.

The third one is uncertainty distribution to describe an uncertain variable. In order to deal with expert's experimental data by uncertainty theory, Liu [6] first built the uncertain statistical method in 2010. Uncertain statistics is based on experts' experimental data collected by designing a questionnaire survey. In uncertain statistics, Liu [6] firstly suggested an empirical uncertainty distribution and proposed a principle of least squares as the method for estimating the unknown parameters. In addition, Wang and Peng [7] proposed a moment method for estimating the unknown parameters in uncertain statistics in 2010. Also, Wang et al. [8] presented uncertain hypothesis testing to detect whether two uncertainty distributions are equal. In order to estimate the uncertainty distribution for an uncertain variable via multiple experts' experimental data, Wang et al. [9] proposed a new method based on the uncertain statistic and Delphi method.

As one of the main contents of uncertainty theory, the concept of uncertain set was first proposed by Liu [10] in 2010 and recast by Liu [11] in 2012. The uncertain set is used to model "unsharp concepts" like "young", "tall", "most", and "about 100 km". The expected value of the uncertain set was defined by Liu [10]. Liu [12] introduced the concepts of variance, entropy, and distance of uncertain sets. More importantly, Liu [11] presented the concept of membership function to describe the uncertain set and provided the operational law of uncertain sets via membership function or inverse membership function. Based on the uncertain set theory, uncertain logic was designed by Liu [12] for dealing with human language by using the truth value formula for uncertain propositions. In addition, a basic uncertain inference rule was proposed by Liu [10]. After that, Gao et al. [13] extended the inference rule to the case with multiple antecedents and multiple if-then rules. Based on the uncertain inference rule, Liu [10] suggested a concept of the uncertain system and then presented an uncertain inference controller as a tool. As a contribution, Peng and Chen [14] proved that an uncertain system is a universal approximator. As a successful application, Gao [15] balanced an inverted pendulum by using the uncertain inference controller.

The uncertain set is mainly characterized by the membership function and operation law. Now, the problem we face is how to get the membership function and how to gain experts' experimental data. In order to determine the membership function of an uncertain set, Liu [12] designed a questionnaire survey to collect experts' experimental data and suggested an empirical membership function. Liu [12] also proposed a principle of least squares as a method for estimating the unknown parameters based on the expert's experimental data. In this paper, we propose a new method aiming at determining the membership function by the combination of the method of uncertain statistics and Delphi method.

The rest of this paper is organized as follows. The next section is intended to introduce some concepts in uncertainty theory as they are needed. Some basic concepts of uncertain statistics are introduced in Section "Experts' Experimental Data". A new method is presented to determine membership function in Section "A New Method". An example is proposed in Section "Estimating the Age Range of the Uncertain Set "young people"" to test the method. Finally, a conclusion is drawn in Section "Conclusions".

Preliminaries

In this section, we will introduce some useful definitions about the uncertain measure, uncertain set, membership function and so on.

Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A number $\mathcal{M}\{\Lambda\}$ indicates the level that Λ will occur. Uncertain measure $\mathcal{M}$ was introduced as a set function satisfying the following axioms (Liu [4]):

Axiom 1. *(Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2. *(Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. In addition, Liu [16] defined the product uncertain measure as follows.

Axiom 4. *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$, for $k = 1, 2, \cdots$, respectively.

The concept of uncertain variable ξ was introduced by Liu [4] as a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. In order to describe an uncertain variable, uncertainty distribution is defined by Liu [4] as

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \forall x \in \Re.$$

Furthermore, the inverse uncertain distribution $\Phi^{-1}(\alpha)$ of ξ was defined by Liu [3]. It plays a crucial role in operations of an uncertain variable.

The expected value is the average value of uncertain variable in the sense of the uncertain measure, and represents the size of the uncertain variable. It was defined by Liu [4] as follows,

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}dx$$

provided that at least one of the two integrals is finite. Assuming ξ has an uncertainty distribution Φ, It was proved by Liu [4] that the expected value of ξ is

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\, dx - \int_{-\infty}^0 \Phi(x)dx.$$

Different from the uncertain variable that is a real-valued function, the uncertain set is a set-valued function on an uncertainty space. A formal definition was given by Liu [10] as follows.

Definition 1. *(Liu [10]) An uncertain set is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers, i.e., for any Borel set B of real number, both of*

$$\{B \subset \xi\} = \{\gamma \in \Gamma | B \subset \xi(\gamma)\}$$

and

$$\{\xi \subset B\} = \{\gamma \in \Gamma | \xi(\gamma) \subset B\}$$

are events.

In order to facilitate the operations, Liu defined the independence of the uncertain set as follows.

Definition 2. *(Liu [17]) The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any Borel set $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \subset B_i\right\}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \subset B_i\right\}$$

where ξ_i^* are arbitrarily chosen from $\left\{\xi_i, \xi_i^c\right\}, i = 1, 2, \cdots, n$, respectively.

Membership function was proposed by Liu to describe the uncertain set.

Definition 3. *(Liu [11]) An uncertain set ξ is said to have a membership function μ if for any Borel set B of real numbers, we have*

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x)$$

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x).$$

Theorem 1. *(Liu [12]) A real-valued function μ is a membership function if and only if*

$$0 \le \mu(x) \le 1.$$

To make the operation easy, Liu proposed the definition of the inverse membership function of the uncertain set and gave some operation laws of independent uncertain sets.

Definition 4. *(Liu [11]) Let ξ be an uncertain set with membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \{x \in \Re | \mu(x) \ge \alpha\}, \forall \alpha \in [0, 1]$$

is called the inverse membership function of ξ. Note that $\mu^{-1}(0)$ is determined by

$$\mu^{-1}(0) = \lim_{\alpha \downarrow 0} \mu^{-1}(\alpha).$$

Sometimes, the set $\mu^{-1}(\alpha)$ is also called the α-cut of μ.

Theorem 2. *(Liu [11]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their union $\xi \cup \eta$ has a membership function*

$$\lambda(x) = \mu(x) \vee \nu(x).$$

Theorem 3. *(Liu [11]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their intersection $\xi \cap \eta$ has a membership function*

$$\lambda(x) = \mu(x) \wedge \nu(x).$$

Theorem 4. *(Liu [11]) Let ξ be an uncertain set with membership function μ. Then, its complement ξ^c has a membership function*

$$\lambda(x) = 1 - \mu(x).$$

Theorem 5. *(Liu [11]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \cdots, \mu_n^{-1}$, respectively. If f is a measurable function, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain set with inverse membership function,

$$\lambda^{-1}(\alpha) = f\left(\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)\right).$$

Remark 1. *The above arithmetic operational law is not equivalent to the extension principle of Zadeh although they coincide with each other in many cases.*

Experts' Experimental Data

Uncertain statistics is based on experts' experimental data rather than historical data. In order to determine the membership function of the uncertain set, Liu [12] designed a questionnaire survey for collecting experts' experimental data. The starting point is to invite one or more domain experts who are asked to complete a questionnaire about the concept of an uncertain set ξ like "about young" individually.

The first step is to ask the domain expert to choose a possible value x that the uncertain set ξ may contain, and then quiz him "how likely does x belong to ξ?". Denote his belief degree by α. An expert's experimental data (x, α) is thus acquired from the domain expert. Repeating the above process, the following expert's experimental data are obtained by the questionnaire.

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n).$$

Let $(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n)$ be the expert's experimental data that meet the following condition

$$x_1 < x_2 < \cdots < x_n. \tag{1}$$

Based on the expert's experimental data, Liu [12] presented the following empirical membership function

$$\mu(x) = \begin{cases} \alpha_i + \frac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i \le x \le x_{i+1}, 1 \le i < n \\ 0, & \text{otherwise} . \end{cases} \tag{2}$$

Assume that a membership function to be determined has a known functional form with one or more unknown parameters like $\mu(x; \theta_1, \theta_2, \cdots, \theta_k)$ where $\theta_1, \theta_2, \cdots, \theta_k$ are unknown parameters. The principle of least squares presented by Liu [12] tells us that the unknown parameters $\theta_i, i = 1, 2, \cdots, k$ should solve the minimization problem

$$\min_{\theta_1,\cdots,\theta_k} \sum_{i=1}^{n} \left(\mu(x; \theta_1, \theta_2, \cdots, \theta_k) - \alpha_i\right)^2 .$$

In the next section, We give a new method to estimate the unknown parameters.

A New Method

Assume that there are m domain experts and each produces an empirical membership function. Then, we may get m empirical membership functions $\mu_1(x), \mu_2(x), \cdots, \mu_m(x)$.

Theorem 6. *Let $\mu_1(x), \mu_2(x), \cdots, \mu_m(x)$ be membership functions. Then*

$$\mu(x) = \omega_1\mu_1(x) + \omega_2\mu_2(x) + \cdots + \omega_m\mu_m(x)$$

is a membership function, where $\omega_i \geq 0, i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} \omega_i = 1$.

Proof. Since $\mu_1(x), \mu_2(x), \cdots, \mu_m(x)$ are membership functions, by Theorem 1, we have $0 \leq \mu_i(x) \leq 1, i = 1, 2, \cdots, m$. Since $\omega_i \geq 0, i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} \omega_i = 1$, then we have $0 \leq \mu(x) = \sum_{i=1}^{m} \omega_i\mu_i(x) \leq 1$. Thus $\mu(x)$ is also a membership function by Theorem 1. □

Remark 2. *$\omega_i, i = 1, 2, \cdots, m$ are convex combination coefficients representing weighs.*

Since its inceptive development by Dalkey and Helmer [18, 19], the Delphi method has been widely accepted as an effective forecasting tool and used in a wide range of applications. It is used to structure a group communication process to facilitate group problem solving and to structure models [20]. The method can also be used as a judgement, decision-aiding, or forecasting tool [21], and can be applied to program planning and administration [22]. The Delphi method can be used when there is incomplete knowledge about a problem or phenomena [22, 23].

The steps of the Delphi method are as follows. Firstly, the participants assess the matters independently and make individual judgement according to individual knowledge and experience. Then, from the second round, the participants are provided with feedback on the previous round so that they can assess the same matters again and make a new judgement about altering their opinions, repeating the above process. Finally, the experts' opinions will tend to be consistent, and a composite survey result is presented.

Let ξ be an uncertain set. Our method is aimed at estimating the membership function $\mu(x)$ of ξ. We first invite m experts to choose some possible values $x_1, x_2, \cdots, x_n$ that the uncertain set ξ may contain (for all experts, their possible values and the number of values can be different). Without loss of generality, we assume $x_1 < x_2 < \cdots < x_n$. Then the procedure can be summarized as follows.

Step 1. Respectively, the m domain experts provide their experimental data (x_{ij}, α_{ij}), where x_{ij} denotes the j-th possible value provided by the i-th expert and α_{ij} denotes the i-th expert's belief degree that x_{ij} belongs to ξ, $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n_i$, respectively.

Step 2. Use the i-th expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ to generate the empirical membership function μ_i of the i-th domain experts, $i = 1, 2, \cdots, m$, respectively.

Step 3. We calculate the number of the possible values of ξ presented by all experts and denote it by n, where the same values from different experts are treated as one. Then, the possible values of ξ are $x_1 \le x_2 \le \cdots \le x_n$, and we continue to compute

$$\bar{\alpha}_j = \frac{1}{m}\sum_{i=1}^{m} \mu_i(x_j),\ j = 1, 2, \cdots, n$$

and

$$d_j = \frac{1}{m}\sum_{i=1}^{m} (\mu_i(x_j) - \bar{\alpha}_j)^2,\ j = 1, 2, \cdots, n.$$

Step 4. If d_j are less than a given level $\varepsilon > 0$ for all j, then go to Step 5. Otherwise, the i-th domain expert receives the summary (for example, the $\bar{\alpha}_j$ obtained in the previous round and the reasons of other experts), and then provide a set of revised experts' experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ for $i = 1, 2, \cdots, m$, go to Step 2.

Step 5. Use the integrated data $(x_1, \bar{\alpha}_1), (x_2, \bar{\alpha}_2), \cdots, (x_n, \bar{\alpha}_n)$ to generate the membership function $\mu(x)$ of the uncertain set ξ.

Estimating the Age Range of the Uncertain Set "young people"

In this section, we will give an example to verify our method that combines uncertain statistics and the Delphi method for estimating the membership function.

The uncertain set is a set-valued function on an uncertainty space and attempts to model "unsharp concepts" that are essentially sets but their boundaries are not sharply described (because of the ambiguity of human language). The concepts that being an uncertain set do not have the property of exclusivity.

In this example, "young people" is considered an uncertain set ξ. In order to obtain the membership function of the uncertain set ξ, six experts are invited to analyze how old are young people. Each expert estimates the age and gives his belief degree on the basis of his knowledge and experience. We will follow the steps above and let the level $\varepsilon = 0.05$. At the same time, we assume that the weight of every expert is $\frac{1}{6}$. The first round experts' experimental data about "young people" are as follows.

$$\begin{aligned}
&E_1: (18, 0), (20, 0.7), (21, 0.9), (22, 1), (26, 1), (27, 0.9), (28, 0.6), (30, 0)\\
&E_2: (17, 0), (18, 0.75), (20, 0.9), (21, 1), (24, 1), (26, 0.7), (28, 0.5), (30, 0)\\
&E_3: (18, 0), (19, 0.7), (20, 0.8), (22, 1), (24, 1), (25, 0.5), (27, 0.2), (28, 0)\\
&E_4: (17, 0), (19, 0.5), (20, 0.9), (21, 1), (25, 1), (27, 0.6), (29, 0.2), (30, 0)\\
&E_5: (15, 0), (18, 0.9), (20, 1), (25, 1), (28, 0.85), (30, 0.7), (32, 0.3), (35, 0)\\
&E_6: (16, 0), (17, 0.4), (18, 0.8), (20, 1), (30, 1), (31, 0.7), (34, 0.5), (35, 0),
\end{aligned}$$

where E_i represents the i-th expert, $i = 1, 2, \cdots, 6$, respectively.

From the data above, six empirical membership functions $\mu_i^{(1)}(x), i = 1, 2, \cdots, 6$ are generated according to (2). On the other hand, the total possible values provided by the six experts are $15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. Then, we compute the corresponding values of $\mu_i^{(1)}(x)$ when $x = 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. See Table 1.

Let

$$\bar{\alpha}_j^{(1)} = \frac{1}{6}\sum_{i=1}^{6} \mu_i^{(1)}(x_j),\ j = 1, 2, \cdots, 19 \tag{3}$$

and

$$d_j^{(1)} = \frac{1}{6}\sum_{i=1}^{6} \left(\mu_i^{(1)}(x_j) - \bar{\alpha}_j^{(1)}\right)^2,\ j = 1, 2, \cdots, 19. \tag{4}$$

By (3), (4), we compute the corresponding $\bar{\alpha}_j^{(1)}$ and $d_j^{(1)}, j = 1, 2, \cdots, 19$. See Table 2.

Because of $d_{16}^{(1)} = 0.1681 > 0.05 = \varepsilon$ in Table 2, we provide each expert with Table 2 as feedback. According to this feedback each expert estimates the scores and his belief degree again. The second round experts' experimental data are as follows:

$$\begin{aligned}
&T_1: (18, 0.2), (20, 0.9), (21, 1), (22, 1), (26, 1), (27, 0.7), (28, 0.5), (30, 0.1)\\
&T_2: (17, 0.1), (18, 0.5), (20, 0.9), (21, 1), (24, 1), (26, 0.7), (28, 0.5), (30, 0.1)\\
&T_3: (18, 0.3), (19, 0.8), (20, 0.9), (22, 1), (24, 1), (25, 0.9), (27, 0.8), (28, 0.3)\\
&T_4: (17, 0.2), (19, 0.8), (20, 0.9), (21, 1), (25, 1), (27, 0.8), (29, 0.5), (30, 0.1)\\
&T_5: (15, 0), (18, 0.6), (20, 1), (25, 1), (28, 0.7), (30, 0.4), (32, 0.2), (35, 0)\\
&T_6: (16, 0.1), (17, 0.3), (18, 0.6), (20, 1), (30, 0.4), (31, 0.3), (34, 0.2), (35, 0).
\end{aligned}$$

Similarly, from the second round data above, six empirical membership functions $\mu_i^{(2)}(x), i = 1, 2, \cdots, 6$ are generated according to (2). On the other hand, the total possible values presented by the six teachers are $15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. Then, we compute the corresponding values of $\mu_i^{(2)}(x)$ when $x = 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. See Table 3.

By (3) and (4), we compute the corresponding $\bar{\alpha}_j^{(2)}$ and $d_j^{(2)}$, where

$$\bar{\alpha}_j^{(2)} = \frac{1}{6}\sum_{i=1}^{6} \mu_i^{(2)}(x_j),\ j = 1, 2, \cdots, 19$$

and

$$d_j^{(2)} = \frac{1}{6}\sum_{l=1}^{6} \left(\mu_l^{(2)}(x_j) - \bar{\alpha}_j^{(2)}\right)^2,\ j = 1, 2, \cdots, 19.$$

See Table 4.

From Table 4,

$$\max_{1\le j\le 19}\left\{d_j^{(2)}\right\} = d_{13}^{(2)} = 0.0517 > 0.05 = \varepsilon.$$

Table 1 The first round belief degree

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\mu_1^{(1)}(x)$	0.0000	0.0000	0.0000	0.0000	0.3500	0.7000	0.9000	1.0000	1.0000	1.0000	1.0000	0.9000	0.6000	0.3000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_2^{(1)}(x)$	0.0000	0.0000	0.0000	0.7500	0.8250	0.9000	1.0000	1.0000	1.0000	0.8500 0.7000	0.6000	0.5000	0.2500	0.0000	0.0000	0.0000	0.0000	0.0000	
$\mu_3^{(1)}(x)$	0.0000	0.0000	0.0000	0.0000	0.7000	0.8000	0.9000	1.0000	1.0000	0.5000	0.3500	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_4^{(1)}(x)$	0.0000	0.0000	0.0000	0.2500	0.5000	0.9000	1.0000	1.0000	1.0000	1.0000	0.8000	0.6000	0.4000	0.2000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_5^{(1)}(x)$	0.0000	0.3000	0.6000	0.9000	0.9500	1.0000	1.0000	1.0000	1.0000	1.0000	0.9500	0.9000	0.8500	0.7750	0.7000	0.5000	0.3000	0.1000	0.0000
$\mu_6^{(1)}(x)$	0.0000	0.0000	0.4000	0.8000	0.9000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.7000	0.6333	0.5000	0.0000

Table 2 The first round mean values and deviations

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\bar{\alpha}_j^{(1)}$	0.0000	0.0500	0.1667	0.4500	0.7042	0.8833	0.9667	1.0000	1.0000	0.8917	0.8000	0.7000	0.5583	0.4208	0.2833	0.2000	0.1556	0.1000	0.0000
$d_j^{(1)}$	0.0000	0.0125	0.0589	0.1433	0.0468	0.0114	0.0022	0.0000	0.0000	0.0337	0.0525	0.0733	0.1037	0.1218	0.1681	0.0833	0.0577	0.0333	0.0000

Table 3 The second round belief degree

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\mu_1^{(2)}(x)$	0.0000	0.0000	0.0000	0.2000	0.5500	0.9000	1.0000	1.0000	0.8500	0.7750	0.7000	0.6000	0.5000	0.3000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_2^{(2)}(x)$	0.0000	0.0000	0.1000	0.5000	0.7000	0.9000	1.0000	1.0000	1.0000	0.8500	0.7000	0.6000	0.5000	0.3000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_3^{(2)}(x)$	0.0000	0.0000	0.0000	0.3000	0.8000	0.9000	0.9500	1.0000	1.0000	0.9000	0.8500	0.8000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_4^{(2)}(x)$	0.0000	0.0000	0.2000	0.5000	0.8000	0.9000	1.0000	1.0000	1.0000	1.0000	0.9000	0.8000	0.6500	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_5^{(2)}(x)$	0.0000	0.2000	0.4000	0.6000	0.8000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9000	0.8000	0.7000	0.5500	0.4000	0.3000	0.2000	0.0667	0.0000
$\mu_6^{(2)}(x)$	0.0000	0.1000	0.3000	0.6000	0.8000	1.0000	0.9400	0.8800	0.7600	0.7000	0.6400	0.5800	0.5200	0.4600	0.4000	0.3000	0.2667	0.2000	0.0000

Table 4 The second round mean values and deviations

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\bar{\alpha}_j^{(2)}$	0.0000	0.0500	0.1667	0.4500	0.7417	0.9333	0.9817	0.9800	0.9350	0.8708	0.7817	0.6967	0.4783	0.3517	0.1333	0.1000	0.0778	0.0444	0.0000
$d_j^{(2)}$	0.0000	0.0058	0.0222	0.0225	0.0087	0.0022	0.0007	0.0020	0.0091	0.0122	0.0110	0.0107	0.0517	0.0337	0.0356	0.0200	0.0125	0.0054	0.0000

in Table 4, we provide each expert with Table 4 as feedback. According to this feedback, each expert estimates the ages and his belief degree again. The third round experts' experimental data are as follows:

$$T_1: (18, 0.5), (20, 0.9), (21, 1), (22, 1), (26, 0.8), (27, 0.7), (28, 0.5), (30, 0.2)$$
$$T_2: (17, 0.2), (18, 0.5), (20, 0.9), (21, 1), (24, 0.9), (26, 0.8), (28, 0.5), (30, 0.2)$$
$$T_3: (18, 0.5), (19, 0.8), (20, 0.9), (22, 1), (24, 1), (25, 0.9), (27, 0.7), (28, 0.5)$$
$$T_4: (17, 0.2), (19, 0.8), (20, 0.9), (21, 1), (25, 0.9), (27, 0.7), (29, 0.4), (30, 0.1)$$
$$T_5: (15, 0), (18, 0.6), (20, 1), (25, 0.9), (28, 0.5), (30, 0.2), (32, 0.1), (35, 0)$$
$$T_6: (16, 0.1), (17, 0.2), (18, 0.6), (20, 1), (30, 0.2), (31, 0.1), (34, 0).$$

Similarly, from the third round data above, six empirical membership functions $\mu_i^{(3)}(x), i = 1, 2, \cdots, 6$ are generated according to (2). On the other hand, the total possible values presented by the six teachers are $15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. Then, we compute the corresponding values of $\mu_i^{(3)}(x)$ when $x = 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28$. See Table 5.

By (3) and (4), we compute the corresponding $\bar{\alpha}_j^{(3)}$ and $d_j^{(3)}$, where

$$\bar{\alpha}_j^{(3)} = \frac{1}{6}\sum_{i=1}^{6} \mu_i^{(3)}(x_j),\ j = 1, 2, \cdots, 19$$

and

$$d_j^{(3)} = \frac{1}{6}\sum_{i=1}^{6} \left(\mu_i^{(3)}(x_j) - \bar{\alpha}_j^{(3)}\right)^2,\ j = 1, 2, \cdots, 19.$$

See Table 6.

From Table 6,

$$\max_{1\le j\le 19}\{d_j^{(3)}\} = d_{13}^{(3)} = 0.0357 < 0.05 = \varepsilon.$$

Let

$$\alpha_j = \frac{1}{6}\sum_{i=1}^{6} \mu_i^{(2)}(x_j), j = 1, 2, \cdots, 19.$$

We get the integrated experts' experimental data

(15, 0), (16, 0.0500), (17, 0.1667), (18, 0.5333), (19, 0.7667), (20, 0.9333), (21, 0.9750), (22, 0.9569), (24, 0.8875), (25, 0.8333), (26, 0.7478), (27, 0.6372), (28, 0.4017), (29, 0.2883), (30, 0.0667), (31, 0.0417), (32, 0.0278), (34, 0.0056), and (35, 0).

Based on experts' experimental data above, an empirical membership function of "young people" is produced and shown by Fig. 1.

Table 5 The third round belief degree

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\mu_1^{(3)}(x)$	0.0000	0.0000	0.0000	0.5000	0.7000	0.9000	1.0000	1.0000	0.9000	0.8500	0.8000	0.7000	0.5000	0.3500	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_2^{(3)}(x)$	0.0000	0.0000	0.2000	0.5000	0.7000	0.9000	1.0000	0.9667	0.9000	0.8500	0.8000	0.6500	0.5000	0.3500	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_3^{(3)}(x)$	0.0000	0.0000	0.0000	0.5000	0.8000	0.9000	0.9500	1.0000	1.0000	0.9000	0.8000	0.7000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_4^{(3)}(x)$	0.0000	0.0000	0.2000	0.5000	0.8000	0.9000	1.0000	0.9750	0.9250	0.9000	0.8000	0.7000	0.5500	0.4000	0.0000	0.0000	0.0000	0.0000	0.0000
$\mu_5^{(3)}(x)$	0.0000	0.2000	0.4000	0.6000	0.8000	1.0000	0.9800	0.9600	0.9200	0.9000	0.7667	0.6333	0.5000	0.3500	0.2000	0.1500	0.1000	0.0333	0.0000
$\mu_6^{(3)}(x)$	0.0000	0.1000	0.2000	0.6000	0.8000	1.0000	0.9200	0.8400	0.6800	0.6000	0.5200	0.4400	0.3600	0.2800	0.2000	0.1000	0.0667	0.0000	0.0000

Table 6 The third round mean values and deviations

x	15	16	17	18	19	20	21	22	24	25	26	27	28	29	30	31	32	34	35
$\bar{\alpha}_j^{(3)}$	0.0000	0.0500	0.1667	0.5333	0.7667	0.9333	0.9750	0.9569	0.8875	0.8333	0.7478	0.6372	0.4017	0.2883	0.0667	0.0417	0.0278	0.0056	0.0000
$d_j^{(3)}$	0.0000	0.0058	0.0189	0.0022	0.0022	0.0022	0.0009	0.0030	0.0097	0.0114	0.0105	0.0085	0.0357	0.0178	0.0089	0.0037	0.0016	0.0002	0.0000

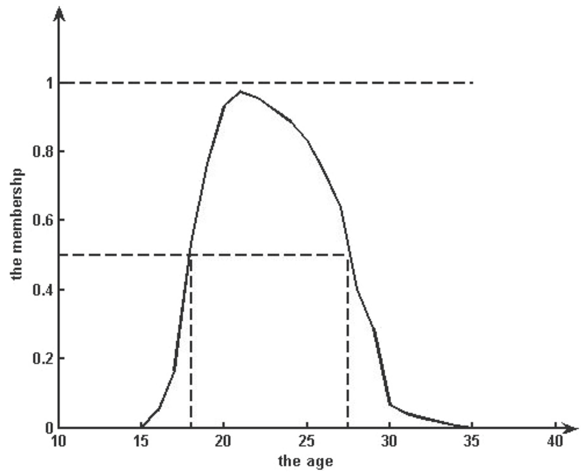

Fig. 1 The empirical membership function of 'young people'

$$\mu(x) = \begin{cases} 0.0500x - 0.7500, & \text{if } 15 \le x \le 16, \\ 0.1167x - 1.8172, & \text{if } 16 \le x \le 17, \\ 0.3666x - 6.0655, & \text{if } 17 \le x \le 18, \\ 0.2334x - 3.6679, & \text{if } 18 \le x \le 19, \\ 0.1666x - 2.3987, & \text{if } 19 \le x \le 20, \\ 0.0417x + 0.0993, & \text{if } 20 \le x \le 21, \\ -0.0181x + 1.3551, & \text{if } 21 \le x \le 22, \\ -0.0347x + 1.7203, & \text{if } 22 \le x \le 24, \\ -0.0542x + 2.1883, & \text{if } 24 \le x \le 25, \\ -0.0855x + 2.9708, & \text{if } 25 \le x \le 26, \\ -0.1106x + 3.6234, & \text{if } 26 \le x \le 27, \\ -0.2355x + 6.9957, & \text{if } 27 \le x \le 28, \\ -0.1134x + 3.5769, & \text{if } 28 \le x \le 29, \\ -0.2216x + 6.7147, & \text{if } 29 \le x \le 30, \\ -0.0250x + 0.8167, & \text{if } 30 \le x \le 31, \\ -0.0139x + 0.4726, & \text{if } 31 \le x \le 32, \\ -0.0111x + 0.3830, & \text{if } 32 \le x \le 34, \\ -0.0056x + 0.1960, & \text{if } 34 \le x \le 35, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We consider people whose ages' membership $\alpha > 0.5$ are "young people". According to Fig. 1, we know the range of young people's ages is 18–27.

Conclusions

Uncertain statistics is a methodology for collecting and interpreting experts' experimental data by uncertainty theory. The method in this paper is a new way to get the empirical membership function when dealing with multiple experts' experimental data. This method depends on each expert's membership function and the Delphi method, and the example shows that this method is effective.

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