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L-fuzzy Fixed Point Theorems for L-fuzzy Mappings via β_{F_L}-admissible with Applications

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Abstract
In this paper, the authors use the idea of β_{F_L}-admissible mappings to prove some L-fuzzy fixed point theorems for a generalized contractive L-fuzzy mappings. Some examples and applications to L-fuzzy fixed points for L-fuzzy mappings in partially ordered metric spaces are also given, to support main findings.

Keywords: L-fuzzy sets, L-fuzzy fixed points, L-fuzzy mappings, β_{F_L}-admissible mappings

AMS Subject Classification: Primary 46S40, Secondary 47H10, 54H25

Introduction

Solving real-world problems becomes apparently easier with the introduction of fuzzy set theory in 1965 by L. A Zadeh [1], as it helps in making the description of vagueness and imprecision clear and more precise. Later in 1967, Goguen [2] extended this idea to L-fuzzy set theory by replacing the interval $[0, 1]$ with a completely distributive lattice L.

In 1981, Heilpern [3] gave a fuzzy extension of Banach contraction principle [4] and Nadler's [5] fixed point theorems by introducing the concept of fuzzy contraction mappings and established a fixed point theorem for fuzzy contraction mappings in a complete metric linear spaces. Frigon and Regan [6] generalized the Heilpern theorem under a contractive condition for 1-level sets of a fuzzy contraction on a complete metric space, where the 1-level sets need not be convex and compact. Subsequently, various generalizations of result in [6] were obtained (see [7–12]). While in 2001, Estruch and Vidal [13] established the existence of a fixed fuzzy point for fuzzy contraction mappings (in the Helpern's sense) on a complete metric space. Afterwards, several authors [11, 14–17] among others studied and generalized the result in [13].

On the other hand, the concept of β-admissible mapping was introduced by Samet et al. [18] for a single-valued mappings and proved the existence of fixed point theorems via this concept, while Asl et al. [19] extended the notion to $\alpha - \psi$-multi-valued mappings. Afterwards, Mohammadi et al. [20] established the notion of β-admissible mapping for the multi-valued mappings (different from the β_*-admissible mapping provided in [19]).

Recently, Phiangsungnoen et al. [21] use the concept of β-admissible defined by Mohammadi et al. [20] to proved some fuzzy fixed point theorems. In 2014, Rashid et al. [22] introduced the notion of β_{F_L}-admissible for a pair of L-fuzzy mappings and utilized it to proved a common L-fuzzy fixed point theorem. The notions of d_L^{∞}-metric and

Hausdorff distances for L-fuzzy sets were introduced by Rashid et al. [23], they presented some fixed point theorems for L-fuzzy set valued-mappings and coincidence theorems for a crisp mapping and a sequence of L-fuzzy mappings. Many researchers have studied fixed point theory in the fuzzy context of metric spaces and normed spaces (see [24–27] and [28–30], respectively).

In this manuscript, the authors developed a new L-fuzzy fixed point theorems on a complete metric space via β_{F_L}-admissble mapping in sense of Mohammadi et al. [20] which is a generalization of main result of Phiangsungnoen et al. [21]. We also construct some examples to support our results and infer as an application, the existence of L-fuzzy fixed points in a complete partially ordered metric space.

Preliminaries

In this section we present some basic definitions and preliminary results which we will used throughout this paper. Let (X,d) be a metric space, $CB(X) = \{A : A \text{ is closed and bounded subsets of } X\}$ and $C(X) = \{A : A \text{ is nonempty compact subsets of } X\}$.

Let $A, B \in CB(X)$ and define

$$
\begin{aligned}
d(x,A) &= \inf_{y\in A} d(x,y),\\
d(A,B) &= \inf_{x\in A, y\in B} d(x,y),\\
p_{\alpha_L}(x,A) &= \inf_{y\in A_{\alpha_L}} d(x,y),\\
p_{\alpha_L}(A,B) &= \inf_{x\in A_{\alpha_L}, y\in B_{\alpha_L}} d(x,y),\\
p(A,B) &= \sup_{\alpha_L} p_{\alpha_L}(A,B),\\
H\left(A_{\alpha_L}, B_{\alpha_L}\right) &= \max\left\{ \sup_{x\in A_{\alpha_L}} d\left(x, B_{\alpha_L}\right), \sup_{y\in B_{\alpha_L}} d\left(y, A_{\alpha_L}\right)\right\},\\
D_{\alpha_L}(A,B) &= H\left(A_{\alpha_L}, B_{\alpha_L}\right),\\
d^{\infty}_{\alpha_L}(A,B) &= \sup_{\alpha_L} D_{\alpha_L}(A,B).
\end{aligned}
$$

Definition 1 *A fuzzy set in X is a function with domain X and range in $[0,1]$. i.e A is a fuzzy set if $A : X \longrightarrow [0,1]$.*

Let $\mathcal{F}(X)$ denotes the collection of all fuzzy subsets of X. If A is a fuzzy set and $x \in X$, then $A(x)$ is called the grade of membership of x in A. The α-level set of A is denoted by $[A]_\alpha$ and is defined as below:

$$
\begin{aligned}
&[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \text{for } \alpha \in (0,1],\\
&[A]_0 = \text{closure of the set } \{x \in X : A(x) > 0\}.
\end{aligned}
$$

Definition 2 *A partially ordered set $(L, \preceq_L)$ is called*

i *a lattice; if $a \vee b \in L, a \wedge b \in L$ for any $a, b \in L$,*

ii *a complete lattice; if $\bigvee A \in L, \bigwedge A \in L$ for any $A \subseteq L$,*

iii *a distributive lattice; if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$,* *iv* *a complete distributive lattice; if* $a \vee (\bigwedge b_i) = \bigwedge_i (a \wedge b_i)$, $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$ *for any* $a, b_i \in L$,

v *a bounded lattice; if it is a lattice and additionally has a top element* 1_L *and a bottom element* 0_L, *which satisfy* $0_L \preceq_L x \preceq_L 1_L$ *for every* $x \in L$.

Definition 3 *An L-fuzzy set A on a nonempty set X is a function* $A : X \longrightarrow L$, *where L is bounded complete distributive lattice with* 1_L *and* 0_L.

Definition 4 (Goguen [2]). *Let L be a lattice, the top and bottom elements of L are* 1_L *and* 0_L *respectively, and if* $a, b \in L$, $a \vee b = 1_L$ *and* $a \wedge b = 0_L$ *then b is a unique complement of a denoted by* $\acute{a}$.

Remark 1 *If* $L = [0, 1]$, *then the L-fuzzy set is the special case of fuzzy sets in the original sense of* Zadeh [1], *which shows that L-fuzzy set is larger.*

Let $\mathcal{F}_L(X)$ denotes the class of all L-fuzzy subsets of X. Define $\mathcal{Q}_L(X) \subset \mathcal{F}_L(X)$ as below:

$$\mathcal{Q}_L(X) = \{A \in \mathcal{F}_L(X) : A_{\alpha_L} \text{ is nonempty and compact}, \alpha_L \in L \backslash \{0_L\}\}.$$

The α_L-level set of an L-fuzzy set A is denoted by A_{α_L} and define as below:

$$A_{\alpha_L} = \{x \in X : \alpha_L \preceq_L A(x)\} \text{ for } \alpha_L \in L \backslash \{0_L\},$$
$$A_{0_L} = \overline{\{x \in X : 0_L \preceq_L A(x)\}}.$$

Where $\overline{B}$ denotes the closure of the set B (Crisp).

For $A, B \in \mathcal{F}_L(X)$, $A \subset B$ if and only if $A(x) \preceq_L B(x)$ for all $x \in X$. If there exists an $\alpha_L \in L \backslash \{0_L\}$ such that $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$, then we define

$$D_{\alpha_L}(A, B) = H(A_{\alpha_L}, B_{\alpha_L}).$$

If $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$ for each $\alpha_L \in L \backslash \{0_L\}$, then we define

$$d_L^{\infty}(A, B) = \sup_{\alpha_L} D_{\alpha_L}(A, B).$$

We note that d_L^{∞} is a metric on $\mathcal{F}_L(X)$ and the completeness of (X, d) implies that $(C(X), H)$ and $(\mathcal{F}_L(X), d_L^{\infty})$ are complete.

Definition 5 *Let X be an arbitrary set, Y be a metric space. A mapping T is called L-fuzzy mapping, if T is a mapping from X to* $\mathcal{F}_L(Y)$*(i.e class of L-fuzzy subsets of Y). An L-fuzzy mapping T is an L-fuzzy subset on* $X \times Y$ *with membership function* $T(x)(y)$. *The function* $T(x)(y)$ *is the grade of membership of y in* $T(x)$.

Definition 6 *Let X be a nonempty set. For* $x \in X$, *we write* $\{x\}$ *the characteristic function of the ordinary subset* $\{x\}$ *of X. The characteristic function of an L-fuzzy set A, is denoted by* χ_{L_A} *and define as below:*

$$\chi_{L_A} = \begin{cases} 0_L & \text{if } x \notin A; \\ 1_L & \text{if } x \in A. \end{cases}$$

Definition 7 *Let* (X, d) *be a metric space and* $T : X \longrightarrow \mathcal{F}_L(X)$. *A point* $z \in X$ *is said to be an L-fuzzy fixed point of T if* $z \in [Tz]_{\alpha_L}$, *for some* $\alpha_L \in L \backslash \{0_L\}$.

Remark 2 *If $\alpha_L = 1_L$, then it is called a fixed point of the L-fuzzy mapping T.*

Definition 8 (Asl et al. [19]). *Let X be a nonempty set. $T : X \longrightarrow 2^X$, where 2^X is a collection of nonempty subsets of X and $\beta : X \times X \longrightarrow [0, \infty)$. We say that T is β_*-admissible if*

$$\text{for } x, y \in X, \beta(x, y) \geq 1 \Longrightarrow \beta_*(Tx, Ty) \geq 1,$$

where

$$\beta_*(Tx, Ty) := \inf\{\beta(a, b) : a \in Tx \text{ and } b \in Ty\}.$$

Definition 9 (Mohammadi et al. [20]). *Let X be a nonempty set. $T : X \longrightarrow 2^X$, where 2^X is a collection of nonempty subsets of X and $\beta : X \times X \longrightarrow [0, \infty)$. We say that T is β-admissible whenever for each $x \in X$ and $y \in Tx$ with $\beta(x, y) \geq 1$, we have $\beta(y, z) \geq 1$ for all $z \in Ty$.*

Remark 3 *If T is β_*-admissible mapping, then T is also β-admissible mapping.*

Example 1 *Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define $T : X \longrightarrow 2^X$ and $\beta : X \times X \longrightarrow [0, \infty)$ by*

$$T(x) = \begin{cases} \left[0, \frac{x}{3}\right], & \text{if } 0 \leq x \leq 1; \\ \left[x^2, \infty\right), & \text{if } x > 1. \end{cases}$$

and

$$\beta(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is β-admissible.

Main Result

L-fuzzy Fixed Point Theorems

Now, we recall some well known results and definitions to be used in the sequel.

Lemma 1 *Let $x \in X, A \in \mathcal{W}_L(X)$,* and *$\{x\}$ be an L-fuzzy set with membership function equal to characteristic function of set $\{x\}$.* If $\{x\} \subset A$, then $p_{\alpha_L}(x, A) = 0$ for $\alpha_L \in L \backslash \{0_L\}$.

Lemma 2 (Nadler [5]). *Let (X, d) be a metric space and $A, B \in CB(X)$. Then for any $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Definition 10 *Let Ψ be the family of non-decreasing functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ where ψ^n is the nth iterate of ψ. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.*

Below, we introduce the concept of β-admissible in the sense of Mohammadi et al. [20] for *L*-fuzzy mappings.

Definition 11 *Let (X, d) be a metric space, $\beta : X \times X \longrightarrow [0, \infty)$ and $T : X \longrightarrow F_L(X)$. A mapping T is said to be β_{F_L}-admissible whenever for each $x \in X$ and $y \in [Tx]_{\alpha_L}$ with $\beta(x, y) \geq 1$, we have $\beta(y, z) \geq 1$ for all $z \in [Ty]_{\alpha_L}$, where $\alpha_L \in L \backslash \{0_L\}$.*

Here, the existence of an L-fuzzy fixed point theorem for some generalized type of contraction L-fuzzy mappings in complete metric spaces is presented.

Theorem 1 *Let (X, d) be a complete metric space, $\alpha_L \in L\backslash\{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ and $\beta : X \times X \longrightarrow [0, \infty)$ such that for all $x, y \in X$,*

$$\beta(x,y)D_{\alpha_L}(Tx, Ty) \leq \psi(\Omega(x,y)) + K \min\left\{p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), p_{\alpha_L}(x, Ty), p_{\alpha_L}(y, Tx)\right\}, \tag{1}$$

where $K \geq 0$ and

$$\Omega(x,y) = \max\left\{d(x,y), p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), \frac{p_{\alpha_L}(x, Ty) + p_{\alpha_L}(y, Tx)}{2}\right\}.$$

If the following conditions hold,

i. if $\{x_n\}$ is a sequence in X so that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to b (n \to \infty)$, then $\beta(x_n, b) \geq 1$,
ii. there exists $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ so that $\beta(x_0, x_1) \geq 1$,
iii. T is β_{F_L}-admissible,
iv. ψ is continuous.

Then T has atleast an L-fuzzy fixed point.

Proof For $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ by condition (ii) we have $\beta(x_0, x_1) \geq 1$. Since $[Tx_0]_{\alpha_L}$ is nonempty and compact, then there exists $x_2 \in [Tx_1]_{\alpha_L}$, such that

$$d(x_1, x_2) = p_{\alpha_L}(x_1, Tx_1) \leq D_{\alpha_L}(Tx_0, Tx_1). \tag{2}$$

By (2) and the fact that $\beta(x_0, x_1) \geq 1$, we have

$$\begin{aligned} d(x_1, x_2) &\leq D_{\alpha_L}(Tx_0, Tx_1) \\ &\leq \beta(x_0, x_1) D_{\alpha_L}(Tx_0, Tx_1) \\ &\leq \psi(\Omega(x_0, x_1)) + K \min\left\{p_{\alpha_L}(x_0, Tx_0), p_{\alpha_L}(x_1, Tx_1),\right. \\ &\quad \left. p_{\alpha_L}(x_0, Tx_1), p_{\alpha_L}(x_1, Tx_0)\right\} \\ &\leq \psi(\Omega(x_0, x_1)) + K \min\left\{p_{\alpha_L}(x_0, x_1), p_{\alpha_L}(x_1, x_2), p_{\alpha_L}(x_0, x_2), 0\right\} \\ &= \psi(\Omega(x_0, x_1)). \end{aligned}$$

Similarly, For $x_2 \in X$, we have $[Tx_2]_{\alpha_L}$ which is nonempty and compact subset of X, then there exists $x_3 \in [Tx_2]_{\alpha_L}$, such that

$$d(x_2, x_3) = p_{\alpha_L}(x_2, Tx_2) \leq D_{\alpha_L}(Tx_1, Tx_2). \tag{3}$$

For $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ with $\beta(x_0, x_1) \geq 1$, by condition (iii) we have $\beta(x_1, x_2) \geq 1$. From (1), (2) and the fact that $\beta(x_1, x_2) \geq 1$, we have

$$\begin{aligned} d(x_2, x_3) &\leq D_{\alpha_L}(Tx_1, Tx_2) \\ &\leq \beta(x_1, x_2) D_{\alpha_L}(Tx_1, Tx_2) \\ &\leq \psi(\Omega(x_1, x_2)) + K \min\left\{p_{\alpha_L}(x_1, Tx_1), p_{\alpha_L}(x_2, Tx_2),\right. \\ &\quad \left. p_{\alpha_L}(x_1, Tx_2), p_{\alpha_L}(x_2, Tx_1)\right\} \\ &\leq \psi(\Omega(x_1, x_2)) + K \min\left\{p_{\alpha_L}(x_1, x_2), p_{\alpha_L}(x_2, x_3), p_{\alpha_L}(x_1, x_3), 0\right\} \\ &= \psi(\Omega(x_1, x_2)). \end{aligned}$$

Continuing in this pattern, a sequence $\{x_n\}$ is obtained such that, for each $n \in \mathbf{N}$, $x_n \in [Tx_{n-1}]_{\alpha_L}$ with $\beta(x_{n-1}, x_n) \geq 1$, we have

$$d(x_n, x_{n+1}) \leq \psi\left(\Omega(x_{n-1}, x_n)\right),$$

where

$$\begin{aligned} \Omega(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), p_{\alpha_L}(x_{n-1}, Tx_{n-1}),\right. \\ &\qquad \left. p_{\alpha_L}(x_n, Tx_n), \frac{p_{\alpha_L}(x_{n-1}, Tx_n) + p_{\alpha_L}(x_n, Tx_{n-1})}{2}\right\} \\ &\leq \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}) \leq \psi\left(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\right), \tag{4}$$

for all $n \in \mathbb{N}$. Now, if there exists $n^* \in \mathbb{N}$ such that $p_{\alpha_L}(x_{n^*}, Tx_{n^*}) = 0$ then by Lemma 1, we have $\{x_{n^*}\} \subset Tx_{n^*}$, that is $x_{n^*} \in [Tx_{n^*}]_{\alpha_L}$ implying that x_{n^*} is an L-fuzzy fixed point of T. So, we suppose that for each $n \in \mathbb{N}$, $p_{\alpha_L}(x_n, Tx_n) > 0$, implying that $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Thus, if $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then by (4) and Definition 10, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

which is a contradiction. Thus, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi\left(d(x_{n-1}, x_n)\right) \\ &\leq \psi\left(\psi\left(d(x_{n-2}, x_{n-1})\right)\right) \\ &\;\vdots \\ &\leq \psi^n d(x_0, x_1). \end{aligned} \tag{5}$$

Next we show that, $\{x_n\}$ is a Cauchy sequence in X. Since $\psi \in \Psi$ and continuous, then there exist $\epsilon > 0$ and a positive integer $h = h(\epsilon)$ such that

$$\sum_{n \geq h} \psi^n d(x_0, x_1) < \epsilon. \tag{6}$$

Let $m > n > h$. By triangular inequality, (5) and (6), we have

$$\begin{aligned} d\left(x_n, x_m\right) &\leq \sum_{k=n}^{m-1} d\left(x_k, x_{k+1}\right) \\ &\leq \sum_{k=n}^{m-1} \psi^k d\left(x_0, x_1\right) \\ &\leq \sum_{n\geq h} \psi^n d\left(x_0, x_1\right) < \epsilon. \end{aligned}$$

Thus, $\{x_n\}$ is Cauchy sequence and since X is complete therefore we have $b \in X$ so that $x_n \to b$ as $n \to \infty$. Now, we show that $b \in [Tb]_{\alpha_L}$. Let us assume the contrary and consider

$$\begin{aligned} d(b, [Tb]_{\alpha_L}) &\leq d(b, x_{n+1}) + d\left(x_{n+1}, [Tb]_{\alpha_L}\right) \\ &\leq d(b, x_{n+1}) + H\left([Tx_n]_{\alpha_L}, [Tb]_{\alpha_L}\right) \\ &\leq d(b, x_{n+1}) + D_{\alpha_L}(Tx_n, Tb) \\ &\leq d(b, x_{n+1}) + \beta(x_n, b) D_{\alpha_L}(Tx_n, Tb) \\ &\leq \psi(\Omega(x_n, b)) + K \min\left\{p_{\alpha_L}(x_n, Tx_n), p_{\alpha_L}(b, Tb), p_{\alpha_L}(x_n, Tb), p_{\alpha_L}(b, Tx_n)\right\} \\ &\leq \psi\left(\max\left\{d(x_n, b), p_{\alpha_L}(x_n, Tx_n), p_{\alpha_L}(b, Tb), \frac{p_{\alpha_L}(x_n, Tb) + p_{\alpha_L}(b, Tx_n)}{2}\right\}\right) \\ &\quad + K \min\left\{p_{\alpha_L}(x_n, Tx_n), p_{\alpha_L}(b, Tb), p_{\alpha_L}(x_n, Tb), p_{\alpha_L}(b, Tx_n)\right\} \\ &= \psi(p_{\alpha_L}(b, Tb)). \end{aligned} \tag{7}$$

Letting $n \to \infty$ in (7), we have

$$\begin{aligned} d\left(b, [Tb]_{\alpha_L}\right) &\leq \psi\left(p_{\alpha_L}(b, Tb)\right) \\ &< p_{\alpha_L}(b, Tb) \\ &= d\left(b, [Tb]_{\alpha_L}\right), \end{aligned}$$

a contraction. Hence,

$$b \in [Tb]_{\alpha_L}, \qquad \alpha_L \in L \backslash \{0_L\}.$$

□

Next, we give an example to support the validity of our result.

Example 2 *Let* $X = [0, 1]$, $d(x, y) = |x - y|$ *for all* $x, y \in X$, *then* (X, d) *is a complete metric space. Let* $L = \{\eta, \kappa, \omega, \tau\}$ *with* $\eta \preceq_L \kappa \preceq_L \tau$, *and* $\eta \preceq_L \omega \preceq_L \tau$, *where* κ *and* ω *are not comparable, therefore* $(L, \preceq_L)$ *is a complete distributive lattice. Define* $T : X \longrightarrow \mathcal{Q}_L(X)$ *as below:*

$$T(x)(t) = \begin{cases} \tau, & if 0 \leq t \leq \frac{x}{6}; \\ \kappa, & if \frac{x}{6} < t \leq \frac{x}{4}; \\ \eta, & if \frac{x}{4} < t \leq \frac{x}{2}; \\ \omega, & if \frac{x}{2} < t \leq 1. \end{cases}$$

For every $x \in X$, $\alpha_L = \tau$ *exists for which*

$$[Tx]_\tau = \left[0, \frac{x}{6}\right].$$

Define $\beta : X \times X \longrightarrow [0, \infty)$ *as below:*

$$\beta(x,y) = \begin{cases} 1, & if x = y; \\ x+1, & if x \neq y. \end{cases}$$

Then, it is easy to see that T *is* β_{F_L}*-admissible. For each* $x, y \in X$ *we have*

$$\begin{aligned} \beta(x,y)D_{\alpha_L}(Tx,Ty) &= \beta(x,y)H\left([Tx]_{\alpha_L}, [Ty]_{\alpha_L}\right) \\ &= \beta(x,y)H\left(\left[0,\frac{x}{6}\right],\left[0,\frac{y}{6}\right]\right) \\ &= \frac{1}{6}\beta(x,y)|x-y| \\ &= \frac{1}{6}\beta(x,y)d(x,y) \\ &< \frac{1}{3}d(x,y) \\ &\leq \psi(\Omega(x,y)) \\ &\quad + K\min\left\{p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), p_{\alpha_L}(x,Ty), p_{\alpha_L}(y,Tx)\right\}. \end{aligned}$$

Where $\psi(t) = \frac{t}{3}$ *for all* $t > 0$ *and* $K \geq 0$*. Conditions (ii) and (iii) of Theorem* 1 *holds obviously. Thus, all the conditions of Theorem* 1 *are satisfied. Hence, there exists a* $0 \in X$ *such that* $0 \in [T0]_\tau$*.*

Below, we introduce the concept of β_*-admissible for L-fuzzy mappings in the sense of Asl et al. [19].

Definition 12 *Let* (X,d) *be a metric space,* $\beta : X \times X \longrightarrow [0,\infty)$ *and* $T : X \longrightarrow F_L(X)$*. A mapping* T *is said to be* $\beta^*_{F_L}$*-admissible if*

$$\text{for } x,y \in X, \alpha_L \in L\backslash\{0_L\}, \beta(x,y) \geq 1 \Longrightarrow \beta^*\left([Tx]_{\alpha_L}, [Ty]_{\alpha_L}\right) \geq 1,$$

where

$$\beta^*\left([Tx]_{\alpha_L}, [Ty]_{\alpha_L}\right) := \inf\left\{\beta(a,b) : a \in [Tx]_{\alpha_L} \text{ and } b \in [Ty]_{\alpha_L}\right\}.$$

Theorem 2 *Let* (X,d) *be a complete metric space,* $\alpha_L \in L\backslash\{0_L\}$ *and* $T : X \longrightarrow \mathcal{Q}_L(X)$ *be an L-fuzzy mapping. Suppose that there exist* $\psi \in \Psi$ *and* $\beta : X \times X \longrightarrow [0,\infty)$ *such that for all* $x, y \in X$,

$$\begin{aligned} \beta(x,y)D_{\alpha_L}(Tx,Ty) &\leq \psi(\Omega(x,y)) \\ &\quad + K\min\left\{p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), p_{\alpha_L}(x,Ty), p_{\alpha_L}(y,Tx)\right\}, \end{aligned}$$

where $K \geq 0$ *and*

$$\Omega(x,y) = \max\left\{d(x,y), p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), \frac{p_{\alpha_L}(x,Ty) + p_{\alpha_L}(y,Tx)}{2}\right\}.$$

If the following conditions hold,

i. *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to u$ as $n \to \infty$, *then* $\beta(x_n, u) \geq 1$,

ii. *there exist* $x_0 \in X$ *and* $x_1 \in [Tx_0]_{\alpha_L}$ *such that* $\beta(x_0, x_1) \geq 1$,

iii. *T is $\beta^*_{F_L}$-admissible,*
iv. *ψ is continuous.*

Then, T has atleast an L-fuzzy fixed point.

Proof By Remark 3 and Theorem 1 the result follows immediately. □

Taking $K = 0$ in Theorem 1 and 2, we obtain the following corollary.

Corollary 1 *Let (X, d) be a complete metric space, $\alpha_L \in L\backslash\{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ and $\beta : X \times X \longrightarrow [0, \infty)$ such that for all $x, y \in X$,*

$$\beta(x,y)D_{\alpha_L}(Tx,Ty) \leq \psi\left(\max\left\{d(x,y), p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), \frac{p_{\alpha_L}(x,Ty)+p_{\alpha_L}(y,Tx)}{2}\right\}\right).$$

If the following conditions hold,

i. *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to u$ as $n \to \infty$, then $\beta(x_n, u) \geq 1$,*
ii. *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ such that $\beta(x_0, x_1) \geq 1$,*
iii. *T is β_{F_L}-admissible (or $\beta^*_{F_L}$-admissible),*
iv. *ψ is continuous.*

Then, T has atleast an L-fuzzy fixed point.

If $\beta(x, y) = 1$ for all $x, y \in X$. Theorem 1 or 2 will reduce to the following result.

Corollary 2 *Let (X, d) be a complete metric space, $\alpha_L \in L\backslash\{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $x, y \in X$,*

$$D_{\alpha_L}(Tx,Ty) \leq \psi(\Omega(x,y)) + K\min\left\{p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), p_{\alpha_L}(x,Ty), p_{\alpha_L}(y,Tx)\right\},$$

where $K \geq 0$ and

$$\Omega(x,y) = \max\left\{d(x,y), p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), \frac{p_{\alpha_L}(x,Ty)+p_{\alpha_L}(y,Tx)}{2}\right\}.$$

Then, T has atleast an L-fuzzy fixed point.

By taking $K = 0$ and $\beta(x, y) = 1$ for all $x, y \in X$ in Theorem 1 or 2, Corollary 1 or 2, we have the following.

Corollary 3 *Let (X, d) be a complete metric space, $\alpha_L \in L\backslash\{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $x, y \in X$,*

$$D_{\alpha_L}(Tx,Ty) \leq \psi\left(\max\left\{d(x,y), p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), \frac{p_{\alpha_L}(x,Ty)+p_{\alpha_L}(y,Tx)}{2}\right\}\right).$$

Then, T has atleast an L-fuzzy fixed point.

Remark 4

i *If we consider $L = [0, 1]$ in Theorem 1 and 2, Corollary 1, 2 and 3 we get Theorem 1, 2 Corollary 2, 4 and 5 of [21] respectively;*

ii *If $\alpha_L = 1_L$ in Theorem 1 and 2, Corollary 1, 2 and 3, then by Remark 2 the L-fuzzy mappings T has atleast a fixed point.*

Applications

In this section, we establish as an application the existence of an L-fuzzy fixed point theorems in complete partially ordered metric spaces.

Below, we present some results which are essential in the remaining part of our work.

Definition 13 *Let X be a nonempty set. Then, $(X, d, \preceq)$ is said to be an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.*

Definition 14 *Let $(X, \preceq)$ be a partially ordered set. Then, $x, y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$ holds.*

For a partially ordered set $(X, \preceq)$, we define

$$\overline{\wedge} := \left\{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\right\}.$$

Definition 15 *A partially ordered set $(X, \preceq)$ is said to satisfy the ordered sequential limit property if $(x_n, x) \in \overline{\wedge}$ for all $n \in \mathbb{N}$, whenever a sequence $x_n \to x$ as $x \to \infty$ and $(x_n, x_{n+1}) \in \overline{\wedge}$ for all $n \in \mathbb{N}$.*

Definition 16 *Let $(X, \preceq)$ be a partially ordered set and $\alpha_L \in L\backslash\{0_L\}$. An L-fuzzy mapping $T : X \longrightarrow \mathcal{Q}_L(X)$ is said to be comparative, if for each $x \in X$ and $y \in [Tx]_{\alpha_L}$ with $(x, y) \in \overline{\wedge}$, we have $(y, z) \in \overline{\wedge}$ for all $z \in [Ty]_{\alpha_L}$.*

Now, the existence of an L-fuzzy fixed point theorem for L-fuzzy mappings in complete partially ordered metric spaces is presented.

Theorem 3 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, $\alpha_L \in L\backslash\{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $(x, y) \in \overline{\wedge}$,*

$$D_{\alpha_L}(Tx, Ty) \leq \psi(\Omega(x, y)) + K \min\{p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), p_{\alpha_L}(x, Ty), p_{\alpha_L}(y, Tx)\}, \quad (8)$$

where $K \geq 0$ and

$$\Omega(x, y) = \max\left\{d(x, y), p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), \frac{p_{\alpha_L}(x, Ty) + p_{\alpha_L}(y, Tx)}{2}\right\}.$$

If the following conditions hold,

I. *X satisfies the order sequential limit property,*
II. *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ such that $(x_0, x_1) \in \overline{\wedge}$,*
III. *T is comparative L-fuzzy mapping,*
IV. *ψ is continuous.*

Then, T has atleast an L-fuzzy fixed point.

Proof Let $\beta : X \times X \longrightarrow [0, \infty)$ be defined as:

$$\beta(x, y) = \begin{cases} 1 \text{ if } (x, y) \in \overline{\wedge}; \\ 0 \text{ if } (x, y) \notin \overline{\wedge}. \end{cases}$$

Now by condition (II), we have $\beta(x_0, x_1) \geq 1$ which implies that condition (ii) of Theorem 1 holds. And since T is comparative L-fuzzy mapping, then condition (iii) of Theorem 1 follows. By (8) and for all $x, y \in X$, we have

$$\begin{aligned} &\beta(x, y) D_{\alpha_L}(Tx, Ty) \\ &\quad \leq \psi(\Omega(x, y)) + K \min\left\{p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), p_{\alpha_L}(x, Ty), p_{\alpha_L}(y, Tx)\right\}. \end{aligned} \tag{9}$$

Condition (i) of Theorem 1 also holds by condition (I). Now that all the hypothesis of Theorem 1 are fulfilled, hence the existence of the L-fuzzy fixed point for L-fuzzy mapping T follows. □

Applying similar technique in the proof of Theorem 3 with Corollary 1, we arrive at the following result.

Corollary 4 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, $\alpha_L \in L \backslash \{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $(x, y) \in \overline{\wedge}$,*

$$D_{\alpha_L}(Tx, Ty) \leq \psi\left(\max\left\{d(x, y), p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), \frac{p_{\alpha_L}(x, Ty) + p_{\alpha_L}(y, Tx)}{2}\right\}\right).$$

If the following conditions hold,

I. X satisfies the order sequential limit property,
II. there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha_L}$ such that $(x_0, x_1) \in \overline{\wedge}$,
III. T is comparative L-fuzzy mapping,
IV. ψ is continuous.

Then, T has at least an L-fuzzy fixed point.

Setting $\beta(x, y) = 1$ for all $(x, y) \in \overline{\wedge}$ and using similar argument in the proof of Theorem 3 with Corollary 2 and 3 we get the followings, respectively.

Corollary 5 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, $\alpha_L \in L \backslash \{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $(x, y) \in \overline{\wedge}$,*

$$D_{\alpha_L}(Tx, Ty) \leq \psi(\Omega(x, y)) + K \min\left\{p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), p_{\alpha_L}(x, Ty), p_{\alpha_L}(y, Tx)\right\},$$

where $K \geq 0$ and

$$\Omega(x, y) = \max\left\{d(x, y), p_{\alpha_L}(x, Tx), p_{\alpha_L}(y, Ty), \frac{p_{\alpha_L}(x, Ty) + p_{\alpha_L}(y, Tx)}{2}\right\}.$$

Then, T has at least an L-fuzzy fixed point.

Corollary 6 *Let $(X, d, \preceq)$ be a complete partially ordered metric space, $\alpha_L \in L \backslash \{0_L\}$ and $T : X \longrightarrow \mathcal{Q}_L(X)$ be an L-fuzzy mapping. Suppose that there exist $\psi \in \Psi$ such that for all $(x, y) \in \overline{\wedge}$,*

$$D_{\alpha_L}(Tx, Ty) \leq \psi\left(\max\left\{d(x,y), p_{\alpha_L}(x,Tx), p_{\alpha_L}(y,Ty), \frac{p_{\alpha_L}(x,Ty)+p_{\alpha_L}(y,Tx)}{2}\right\}\right).$$

Then, T has at least an L-fuzzy fixed point.

Remark 5

i. *If we consider* $L=[0,1]$ *in Theorem* 3 *and Corollary* 4 *above, we get Theorem* 3 *and Corollary* 7 *of* [21], *respectively;*
ii. *If* $\alpha_L = 1_L$ *in Theorem* 3, *Corollary* 4, 5 *and* 6, *then by Remark* 2 *the L-fuzzy mappings T has at least a fixed point.*

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